Introduction to Eigenvalue Solver for ITG and Gyrofluid Simulation using BOUT++ GLF code in Core Region

S. S. Kim, H. Jhang, P. H. Diamond, T. Rhee, G. Y. Park

WCI Center for Fusion Theory, NFRI, Korea

X. Q. Xu, P. W. Xi, A. Dimits, M. Umansky

LLNL

BOUT++ 2013 Workshop

Introduction to

Eigenvalue Solver for ITG

Governing equations for ITG

(Ottaviani et. al., PoP '99)

- Vorticity equation

$$\frac{\partial \Omega_i}{\partial t} + \mathbf{V}_E \cdot \nabla \Omega_i = -n \nabla_\parallel V_\parallel + n\left(\mathbf{V}_E + \mathbf{V}_{p_i}\right) \cdot \left(\boldsymbol{\kappa} + \nabla \ln B\right) + n \mathbf{V}_{p_i} \cdot \nabla \left(\frac{n_1 - \Omega_i}{n}\right) - \mathbf{V}_E \cdot \nabla n_{eq} + D_c \Omega_i$$

where $\Omega_i = n_1 - n_{pol}$: generalized vorticity, $n_{pol} = \frac{ne}{m_i \omega_{ci}^2} \nabla_\perp^2 \varphi$, $n_1 = n_{eq} \frac{e\varphi_1}{T_e}$: adiabatic response

$$\mathbf{V}_E = \frac{c}{B} \mathbf{b} \times \nabla \varphi, \quad \mathbf{V}_{p_i} = \frac{c}{neB} \mathbf{b} \times \nabla p_i$$

- Ion temperature equation

$$\frac{\partial T_i}{\partial t} + \mathbf{V}_E \cdot \nabla T_i = -\frac{2}{3} T_i \nabla_\parallel V_\parallel + D_c T_i + D_{glf} T_i$$

- Ion parallel velocity equation

$$\frac{\partial V_\parallel}{\partial t} + \mathbf{V}_E \cdot \nabla V_\parallel = -\frac{e}{m_i} \nabla_\parallel \varphi - \frac{1}{m_i n} \nabla_\parallel p_i + D_c V_\parallel$$

where $D_c F = \mu_1 \nabla_\perp^2 F - \mu_2 \nabla_\perp^4 F$: viscous damping, $D_{glf} T_i = -\sqrt{\frac{8 T_{i,eq}}{\pi m_i}} \left|\nabla_\parallel\right| T_{i1}$: Landau damping.

Linearized equations

$$\frac{\partial\tilde{\Omega}_i}{\partial\bar{t}} = -\hat{n}_i\hat{\nabla}_\parallel\tilde{V}_{i\parallel} + i\left(\hat{n}_i\omega_{di}\tilde{\varphi} + \omega_{di}\tilde{p}_i\right) + \left[\frac{\hat{p}_i}{\rho_*}, \tilde{\nabla}_\perp^2\tilde{\varphi}\right] - \left[\tilde{\Psi}, \frac{\hat{n}_i}{\rho_*}\right] + \overline{D}_c\tilde{\Omega}_i$$

$$\frac{\partial\tilde{T}_i}{\partial\bar{t}} + \left[\tilde{\Psi}, \frac{\hat{T}_i}{\rho_*}\right] = -\frac{2}{3}\hat{T}_i\nabla_\parallel\tilde{V}_{i\parallel} + \overline{D}_c\tilde{T}_i + \overline{D}_{glf}\tilde{T}_i$$

$$\frac{\partial\tilde{V}_{i\parallel}}{\partial\bar{t}} = -\hat{\nabla}_\parallel\tilde{\varphi} - \frac{1}{\hat{n}_i}\hat{\nabla}_\parallel\tilde{p}_i + \overline{D}_c\tilde{V}_{i\parallel}$$

where

$$\tilde{\Omega}_i = \tilde{n}_i - \hat{n}_i\tilde{\nabla}_\perp^2\tilde{\varphi}, \quad \tilde{n}_i = \frac{\hat{n}_i}{\hat{T}_e}\tilde{\varphi},$$

$$\tilde{\Psi} = \Gamma_0^{1/2}\tilde{\varphi} = \left(1 - \frac{1}{2}\tilde{\nabla}_\perp^2\right)^{-1}\tilde{\varphi}$$

is gyroaveraged potential.

following normalizations are used,

$$F = F_{eq} + F_1, \quad F = \left(\Omega, V_\parallel, n, T, \varphi\right), \quad F_0 = \left(n_0, c_{s0}, n_0, T_0, \frac{T_0}{e}\right)$$

$$\hat{F} = \frac{F_{eq}}{F_0}, \quad \tilde{F} = \frac{F_1}{\rho_* F_0}, \quad \tilde{\nabla}_\perp = \rho_* a\nabla_\perp, \quad \hat{\nabla}_\parallel = a\nabla_\parallel, \quad \rho = \frac{r}{a}, \quad \bar{t} = \frac{t}{t_0}$$

$$\rho_* = \frac{\rho_{s0}}{a}, \quad \rho_{s0} = \frac{c_{s0}}{\omega_{c0}}, \quad c_{s0} = \sqrt{\frac{T_0}{m_i}}, \quad \omega_{c0} = \frac{e_i B_0}{m_i c}, \quad t_0 = \frac{a}{c_{s0}}$$

$$[f, g] \equiv \rho_{s0}^2\mathbf{b}\times\nabla f\cdot\nabla g = \frac{\rho_*^2}{\rho}\left(\frac{\partial f}{\partial\rho}\frac{\partial g}{\partial\theta} - \frac{\partial g}{\partial\rho}\frac{\partial f}{\partial\theta}\right)$$

$$i\omega_{di}\tilde{F} \equiv 2\rho_{s0}a\mathbf{b}\times\nabla\tilde{F}\cdot\nabla\ln B = 2\rho_*\frac{a}{R}\left(\frac{1}{\rho}\cos\theta\frac{\partial\tilde{F}}{\partial\theta} + \sin\theta\frac{\partial\tilde{F}}{\partial\rho}\right)$$

Note: F_0 is constant (not the value on magnetic axis)

$$\frac{\partial\tilde{\Omega}_i}{\partial\bar{t}} = -\hat{n}_i\hat{\nabla}_{\parallel}\tilde{V}_{i\parallel} + i\left(\hat{n}_i\omega_{di}\tilde{\varphi} + \omega_{di}\tilde{p}_i\right) + \left[\frac{\hat{p}_i}{\rho_*}, \tilde{\nabla}_{\perp}^2\tilde{\varphi}\right] + \left[\frac{\hat{n}_i}{\rho_*}, \tilde{\Psi}\right] + \overline{D}_c\tilde{\Omega}_i$$

$$\Rightarrow \quad \omega M\tilde{\varphi} = \hat{n}_i k_{\parallel}\tilde{V}_{i\parallel} - a_W\tilde{\varphi} - b_W\tilde{T}_i + G_{\hat{p}_i}L\tilde{\varphi} + G_{\hat{n}_i}J_0\tilde{\varphi} + i\overline{D}_c M\tilde{\varphi}$$

$$\frac{\partial\tilde{T}_i}{\partial\bar{t}} = \left[\frac{\hat{T}_i}{\rho_*}, \tilde{\Psi}\right] - \frac{2}{3}\hat{T}_i\nabla_{\parallel}\tilde{V}_{i\parallel} + \overline{D}_c\tilde{T}_i + \overline{D}_{glf}\tilde{T}_i$$

$$\Rightarrow \quad \omega\tilde{T}_i = G_{\hat{T}_i}J_0\tilde{\varphi}_i + \frac{2}{3}\hat{T}_i k_{\parallel}\tilde{V}_{i\parallel} + i\overline{D}_c\tilde{T}_i + i\overline{D}_{glf}\tilde{T}_i$$

$$\frac{\partial\tilde{V}_{i\parallel}}{\partial\bar{t}} = -\hat{\nabla}_{\parallel}\tilde{\varphi} - \frac{1}{\hat{n}_i}\hat{\nabla}_{\parallel}\tilde{p}_i + \overline{D}_c\tilde{V}_{i\parallel}$$

$$\Rightarrow \quad \omega\tilde{V}_{i\parallel} = \left(1 + \frac{\hat{T}_i}{\hat{T}_e}\right)k_{\parallel}\tilde{\varphi} + k_{\parallel}\tilde{T}_i + i\overline{D}_c\tilde{V}_{i\parallel}$$

where we introduce

$$\frac{\partial}{\partial\bar{t}} \equiv -i\omega, \qquad \tilde{\nabla}_{\perp}^2\tilde{\varphi} \equiv L\tilde{\varphi}, \qquad \hat{\nabla}_{\parallel} \equiv -ik_{\parallel}$$

$$\tilde{\Omega}_i = \left(\frac{\hat{n}_i}{\hat{T}_e} - \hat{n}_i\tilde{\nabla}_{\perp}^2\right)\tilde{\varphi} \equiv M\tilde{\varphi}$$

$$\tilde{\Psi} = \left(1 - \frac{1}{2}\tilde{\nabla}_{\perp}^2\right)^{-1}\tilde{\varphi} \equiv J_0\tilde{\varphi}$$

$$\left[\frac{\hat{F}}{\rho_*}, \quad\right] = \rho_{s0}^2\mathbf{b}\times\nabla\frac{\hat{F}}{\rho_*}\cdot\nabla \equiv -iG_{\hat{F}}$$

$$i\left(\hat{n}_i\omega_{di}\tilde{\varphi} + \omega_{di}\tilde{p}_i\right) \equiv i\left(a_W\tilde{\varphi} + b_W\tilde{T}_i\right)$$

and use

$$\tilde{p}_i = \frac{\hat{p}_i}{\hat{T}_e}\tilde{\varphi}_i + \hat{n}_i\tilde{T}_i.$$

$$\frac{\partial\tilde{\Omega}_i}{\partial\bar{t}} = -\hat{n}_i\hat{\nabla}_\parallel\tilde{V}_{i\parallel} + i\left(\hat{n}_i\omega_{di}\tilde{\varphi} + \omega_{di}\tilde{p}_i\right) + \left[\frac{\hat{p}_i}{\rho_*}, \tilde{\nabla}_\perp^2\tilde{\varphi}\right] + \left[\frac{\hat{n}_i}{\rho_*}, \tilde{\Psi}\right] + \overline{D}_c\tilde{\Omega}_i$$

$$\Rightarrow \quad \omega M\tilde{\varphi} = \hat{n}_i k_\parallel \tilde{V}_{i\parallel} - a_W\tilde{\varphi} - b_W\tilde{T}_i + G_{\hat{p}_i} L\tilde{\varphi} + G_{\hat{n}_i} J_0\tilde{\varphi} + i\overline{D}_c M\tilde{\varphi}$$

$$\Rightarrow \quad \boxed{\omega\tilde{\varphi} = M^{-1}\left(-a_W + G_{\hat{p}_i} L + G_{\hat{n}_i} J_0 + i\overline{D}_c M\right)\tilde{\varphi} - M^{-1}b_W\tilde{T}_i + M^{-1}\hat{n}_i k_\parallel\tilde{V}_{i\parallel}}$$

$$\frac{\partial\tilde{T}_i}{\partial\bar{t}} = \left[\frac{\hat{T}_i}{\rho_*}, \tilde{\Psi}\right] - \frac{2}{3}\hat{T}_i\nabla_\parallel\tilde{V}_{i\parallel} + \overline{D}_c\tilde{T}_i + \overline{D}_{glf}\tilde{T}_i$$

$$\Rightarrow \quad \omega\tilde{T}_i = G_{\hat{T}_i} J_0\tilde{\varphi}_i + \frac{2}{3}\hat{T}_i k_\parallel\tilde{V}_{i\parallel} + i\overline{D}_c\tilde{T}_i + i\overline{D}_{glf}\tilde{T}_i$$

$$\Rightarrow \quad \boxed{\omega\tilde{T}_i = G_{\hat{T}_i} J_0\tilde{\varphi}_i + \left(i\overline{D}_c + i\overline{D}_{glf}\right)\tilde{T}_i + \frac{2}{3}\hat{T}_i k_\parallel\tilde{V}_{i\parallel}}$$

$$\frac{\partial\tilde{V}_{i\parallel}}{\partial\bar{t}} = -\hat{\nabla}_\parallel\tilde{\varphi} - \frac{1}{\hat{n}_i}\hat{\nabla}_\parallel\tilde{p}_i + \overline{D}_c\tilde{V}_{i\parallel}$$

$$\Rightarrow \quad \boxed{\omega\tilde{V}_{i\parallel} = \left(1 + \frac{\hat{T}_i}{\hat{T}_e}\right)k_\parallel\tilde{\varphi} + k_\parallel\tilde{T}_i + i\overline{D}_c\tilde{V}_{i\parallel}}$$

where we introduce

$$\frac{\partial}{\partial\bar{t}} \equiv -i\omega, \qquad \tilde{\nabla}_\perp^2\tilde{\varphi} \equiv L\tilde{\varphi}, \qquad \hat{\nabla}_\parallel \equiv -ik_\parallel$$

$$\tilde{\Omega}_i = \left(\frac{\hat{n}_i}{\hat{T}_e} - \hat{n}_i\tilde{\nabla}_\perp^2\right)\tilde{\varphi} \equiv M\tilde{\varphi}$$

$$\tilde{\Psi} = \left(1 - \frac{1}{2}\tilde{\nabla}_\perp^2\right)^{-1}\tilde{\varphi} \equiv J_0\tilde{\varphi}$$

$$\left[\frac{\hat{F}}{\rho_*},\ \right] = \rho_{s0}^2\mathbf{b}\times\nabla\frac{\hat{F}}{\rho_*}\cdot\nabla \equiv -iG_{\hat{F}}$$

$$i\left(\hat{n}_i\omega_{di}\tilde{\varphi} + \omega_{di}\tilde{p}_i\right) \equiv i\left(a_W\tilde{\varphi} + b_W\tilde{T}_i\right)$$

and use

$$\tilde{p}_i = \frac{\hat{p}_i}{\hat{T}_e}\tilde{\varphi}_i + \hat{n}_i\tilde{T}_i.$$

$$\frac{\partial\tilde{\Omega}_i}{\partial\bar{t}} = -\hat{n}_i\hat{\nabla}_\parallel\tilde{V}_{i\parallel} + i\left(\hat{n}_i\omega_{di}\tilde{\varphi} + \omega_{di}\tilde{p}_i\right) + \left[\frac{\hat{p}_i}{\rho_*}, \tilde{\nabla}_\perp^2\tilde{\varphi}\right] + \left[\frac{\hat{n}_i}{\rho_*}, \tilde{\Psi}\right] + \overline{D}_c\tilde{\Omega}_i$$

$$\Rightarrow \quad \omega M\tilde{\varphi} = \hat{n}_i k_\parallel \tilde{V}_{i\parallel} - a_W\tilde{\varphi} - b_W\tilde{T}_i + G_{\hat{p}_i}L\tilde{\varphi} + G_{\hat{n}_i}J_0\tilde{\varphi} + i\overline{D}_c M\tilde{\varphi}$$

$$\Rightarrow \quad \boxed{\omega\tilde{\varphi} = M^{-1}\left(-a_W + G_{\hat{p}_i}L + G_{\hat{n}_i}J_0 + i\overline{D}_c M\right)\tilde{\varphi} - M^{-1}b_W\tilde{T}_i + M^{-1}\hat{n}_i k_\parallel \tilde{V}_{i\parallel}}$$

$$\frac{\partial\tilde{T}_i}{\partial\bar{t}} = \left[\frac{\hat{T}_i}{\rho_*}, \tilde{\Psi}\right] - \frac{2}{3}\hat{T}_i\nabla_\parallel\tilde{V}_{i\parallel} + \overline{D}_c\tilde{T}_i + \overline{D}_{glf}\tilde{T}_i$$

$$\Rightarrow \quad \omega\tilde{T}_i = G_{\hat{T}_i}J_0\tilde{\varphi}_i + \frac{2}{3}\hat{T}_i k_\parallel \tilde{V}_{i\parallel} + i\overline{D}_c\tilde{T}_i + i\overline{D}_{glf}\tilde{T}_i$$

$$\Rightarrow \quad \boxed{\omega\tilde{T}_i = G_{\hat{T}_i}J_0\tilde{\varphi}_i + \left(i\overline{D}_c + i\overline{D}_{glf}\right)\tilde{T}_i + \frac{2}{3}\hat{T}_i k_\parallel \tilde{V}_{i\parallel}}$$

$$\frac{\partial\tilde{V}_{i\parallel}}{\partial\bar{t}} = -\hat{\nabla}_\parallel\tilde{\varphi} - \frac{1}{\hat{n}_i}\hat{\nabla}_\parallel\tilde{p}_i + \overline{D}_c\tilde{V}_{i\parallel}$$

$$\Rightarrow \quad \boxed{\omega\tilde{V}_{i\parallel} = \left(1 + \frac{\hat{T}_i}{\hat{T}_e}\right)k_\parallel\tilde{\varphi} + k_\parallel\tilde{T}_i + i\overline{D}_c\tilde{V}_{i\parallel}}$$

Those operators can be represented as matrices in a functional space.

where we introduce

$$\frac{\partial}{\partial\bar{t}} \equiv -i\omega, \qquad \tilde{\nabla}_\perp^2\tilde{\varphi} \equiv L\tilde{\varphi}, \qquad \hat{\nabla}_\parallel \equiv -ik_\parallel$$

$$\tilde{\Omega}_i = \left(\frac{\hat{n}_i}{\hat{T}_e} - \hat{n}_i\tilde{\nabla}_\perp^2\right)\tilde{\varphi} \equiv M\tilde{\varphi}$$

$$\tilde{\Psi} = \left(1 - \frac{1}{2}\tilde{\nabla}_\perp^2\right)^{-1}\tilde{\varphi} \equiv J_0\tilde{\varphi}$$

$$\left[\frac{\hat{F}}{\rho_*},\ \right] = \rho_{s0}^2\mathbf{b}\times\nabla\frac{\hat{F}}{\rho_*}\cdot\nabla \equiv -iG_{\hat{F}}$$

$$i\left(\hat{n}_i\omega_{di}\tilde{\varphi} + \omega_{di}\tilde{p}_i\right) \equiv i\left(a_W\tilde{\varphi} + b_W\tilde{T}_i\right)$$

and use

$$\tilde{p}_i = \frac{\hat{p}_i}{\hat{T}_e}\tilde{\varphi}_i + \hat{n}_i\tilde{T}_i.$$

Final form of eigenvalue equation

$$\omega\tilde{\varphi} = A_{11}\tilde{\varphi} + A_{12}\tilde{T}_i + A_{13}\tilde{V}_{i\parallel}$$

$$\omega\tilde{T}_i = A_{21}\tilde{\varphi} + A_{22}\tilde{T}_i + A_{23}\tilde{V}_{i\parallel}$$

$$\omega\tilde{V}_{i\parallel} = A_{31}\tilde{\varphi} + A_{32}\tilde{T}_i + A_{33}\tilde{V}_{i\parallel}$$

$$\Rightarrow \quad \boxed{\omega\mathbf{F} = \mathbf{A}\cdot\mathbf{F}}$$

$$\mathbf{F} = \begin{pmatrix} \tilde{\boldsymbol{\varphi}} \\ \tilde{\mathbf{T}}_{\mathbf{i}} \\ \tilde{\mathbf{V}}_{\mathbf{i}\parallel} \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{A}_{13} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \mathbf{A}_{23} \\ \mathbf{A}_{31} & \mathbf{A}_{32} & \mathbf{A}_{33} \end{pmatrix}$$

$$\tilde{\boldsymbol{\varphi}} = \sum_k \tilde{\varphi}_k |k\rangle = \begin{pmatrix} \tilde{\varphi}_1 \\ \tilde{\varphi}_2 \\ \tilde{\varphi}_3 \\ \vdots \end{pmatrix}, \quad \cdots$$

$|k\rangle$ is a basis function

where

$$A_{11} = M^{-1}\left(-a_W + G_{\hat{p}_i} L + G_{\hat{n}_i} J_0 + i\overline{D}_c M\right), \quad A_{12} = -M^{-1} b_w, \quad A_{13} = M^{-1}\hat{n}_i k_\parallel$$

$$A_{21} = G_{\hat{T}_i} J_0, \quad A_{22} = i\overline{D}_c + i\overline{D}_{glf}, \quad A_{23} = \frac{2}{3}\hat{T}_i k_\parallel$$

$$A_{31} = \left(1 + \frac{\hat{T}_i}{\hat{T}_e}\right) k_\parallel, \quad A_{32} = k_\parallel, \quad A_{33} = i\overline{D}_c$$

Basis functions

Field F can represented as $F(\rho,\theta,\zeta,t)=e^{-i\omega t}\sum_{k=mnp}F_k W_k(\rho)e^{i(m\theta-n\zeta)}$ in concentric-circular geometry :

$$\mathbf{F}=|F\rangle=e^{-i\omega t}\sum_k F_k|k\rangle, \quad \boxed{|k\rangle=|mnp\rangle=W_{mnp}(\rho)e^{i(m\theta-n\zeta)}},$$

inner product :

$$\langle k|F\rangle\equiv\frac{1}{4\pi^2}\int_0^1\rho d\rho\int_0^{2\pi}d\theta\int_0^{2\pi}d\zeta W_{mnp}(\rho)e^{-i(m\theta-n\zeta)}F, \qquad \langle k|k'\rangle=\delta_{kk'}$$

- Radial basis function

For Hermite basis ,

$$W_{mnp}(\rho)=\frac{1}{\sqrt{2\rho w_{mn}\nu_p}}H_p(x)e^{-x^2/2}$$

where $x=\dfrac{\rho-\rho_{mn}}{w_{mn}}$ $\left(\rho_{mn}\text{ is radial position of rational surface satisfying } q=\dfrac{m}{n}\right)$, $\quad \nu_p=2^{p/2}\Gamma(p+1)^{1/2}\pi^{1/4}$

For Bessel basis,

$$W_{mnp}(\rho)=\frac{\sqrt{2}}{J_{m+1}(\alpha_{mp})}J_m(\alpha_{mp}\rho) \quad \text{where } J_m(\alpha_{mp})=0$$

Matrix (M, J0, Dc) involving Laplacian L

For Bessel basis,

$$L_{kk'} = \langle k | \tilde{\nabla}_\perp^2 | k' \rangle = \rho_*^2 \langle k \| \left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} - \frac{m'^2}{\rho^2} \right) \| k' \rangle = -k_\perp^2 \delta_{kk'}$$

where $k_\perp = \rho_* \alpha_{mp}$

$$M_{kk'} \approx \left. \frac{\hat{n}_i}{\hat{T}_e} \right|_{\rho=\rho_{mn}} \delta_{kk'} - \hat{n}_i \big|_{\rho=\rho_{mn}} L_{kk'}$$

$$J_{0,kk'} = \left(1 - \frac{1}{2} L_{kk'} \right)^{-1}$$

$$\overline{D}_{c,kk'} = \overline{\mu}_1 L_{kk'} - \overline{\mu}_2 L_{kk'}^2$$

$$L \tilde{\varphi} = \tilde{\nabla}_\perp^2 \tilde{\varphi}$$

$$M \tilde{\varphi} = \left(\frac{\hat{n}_i}{\hat{T}_e} - \hat{n}_i \tilde{\nabla}_\perp^2 \right) \tilde{\varphi} = \tilde{\Omega}_i$$

$$J_0 \tilde{\varphi} = \left(1 - \frac{1}{2} \tilde{\nabla}_\perp^2 \right)^{-1} \tilde{\varphi} = \tilde{\Psi}$$

$$\overline{D}_c F = \overline{\mu}_1 \tilde{\nabla}_\perp^2 F - \overline{\mu}_2 \tilde{\nabla}_\perp^4 F$$

Matrix ($k_{||}$, D_{glf}) involving parallel wavenumber

$$\hat{F}k_{\|,kk'} = i\left\langle k\left|\hat{F}\hat{\nabla}_{\|}\right|k'\right\rangle = \delta_{mm'}\delta_{nn'}\frac{a}{R}\int_0^1 \hat{F}(\rho)\left(\frac{m}{q(\rho)}-n\right)W_{mnp}(\rho)W_{mnp'}(\rho)\rho d\rho$$

$$\overline{D}_{glf,kk'} = -\left\langle k\left|\sqrt{\frac{8\hat{T}_i}{\pi}}\left|\hat{\nabla}_{\|}\right|\right|k'\right\rangle = -\delta_{mm'}\delta_{nn'}\sqrt{\frac{8}{\pi}}\frac{a}{R}\int_0^1 \sqrt{\hat{T}_i(\rho)}\left|\frac{m}{q(\rho)}-n\right|W_{mnp}(\rho)W_{mnp'}(\rho)\rho d\rho$$

Matrix (G) for ExB and diamagnetic drifts

$$G_{\hat{F}}\tilde{\varphi} = i\left[\frac{\hat{F}}{\rho_*},\tilde{\varphi}\right]:$$

$$G_{\hat{F},kk'} = i\left\langle k\left|\left[\frac{\hat{F}}{\rho_*},\quad\right]\right|k'\right\rangle = i\rho_*\left\langle k\left|\frac{1}{\rho}\frac{\partial\hat{F}}{\partial\rho}\frac{\partial}{\partial\theta}\right|k'\right\rangle = -\delta_{mm'}\delta_{nn'}\rho_* m\int_0^1 \frac{\partial\hat{F}}{\partial\rho}W_{mnp}(\rho)W_{mnp'}(\rho)d\rho$$

Matrix (a_W, b_W) involving curvature

$$\hat{\beta}\omega_{di}\hat{\alpha}\left|k'\right\rangle = -2i\rho_* \frac{a}{R}\hat{\beta}\left(\frac{1}{\rho}\cos\theta\frac{\partial}{\partial\theta} + \sin\theta\frac{\partial}{\partial\rho}\right)\hat{\alpha}\left|k'\right\rangle$$

$$= \rho_* \frac{a}{R}\left[\left(\frac{m'}{\rho}\hat{\beta}\hat{\alpha}W_{k'} + \hat{\beta}\frac{\partial\hat{\alpha}W_{k'}}{\partial\rho}\right)e^{i(m'-1)\theta} + \left(\frac{m'}{\rho}\hat{\beta}\hat{\alpha}W_{k'} - \hat{\beta}\frac{\partial\hat{\alpha}W_{k'}}{\partial\rho}\right)e^{i(m'+1)\theta}\right]e^{-in'\zeta}$$

$$\Rightarrow \quad \left\langle k\left|\hat{\beta}\omega_{di}\hat{\alpha}\right|k'\right\rangle = \rho_*\frac{a}{R}\left[\begin{aligned}&\left\{(m+1)\int_0^1 \hat{\beta}\hat{\alpha}W_{mnp}W_{m+1,n,p'}d\rho + \int_0^1 \hat{\beta}W_{mnp}\frac{\partial\hat{\alpha}W_{m+1,n,p'}}{\partial\rho}\rho d\rho\right\}\delta_{m+1,m'}\delta_{nn'}\\ &+\left\{(m-1)\int_0^1 \hat{\beta}\hat{\alpha}W_{mnp}W_{m-1,n,p'}d\rho - \int_0^1 \hat{\beta}W_{mnp}\frac{\partial\hat{\alpha}W_{m-1,n,p'}}{\partial\rho}\rho d\rho\right\}\delta_{m-1,m'}\delta_{nn'}\end{aligned}\right]$$

$$\Downarrow$$

$$a_{W,kk'} = \left\langle k\left|\left(\hat{n}_i\omega_{di} + \omega_{di}\frac{\hat{p}_i}{\hat{T}_e}\right)\right|k'\right\rangle = a_W^+(k,p')\delta_{m+1,m'}\delta_{nn'} + a_W^-(k,p')\delta_{m-1,m'}\delta_{nn'}$$

$$b_{W,kk'} = \left\langle k\left|\omega_{di}\hat{n}_i\right|k'\right\rangle = b_W^+(k,p')\delta_{m+1,m'}\delta_{nn'} + b_W^-(k,p')\delta_{m-1,m'}\delta_{nn'}$$

$$a_W\tilde{\varphi} = \hat{n}_i\omega_{di}\tilde{\varphi} + \omega_{di}\frac{\hat{p}_i}{\hat{T}_e}\tilde{\varphi}$$

$$b_W\tilde{T}_i = \omega_{di}\hat{n}_i\tilde{T}_i$$

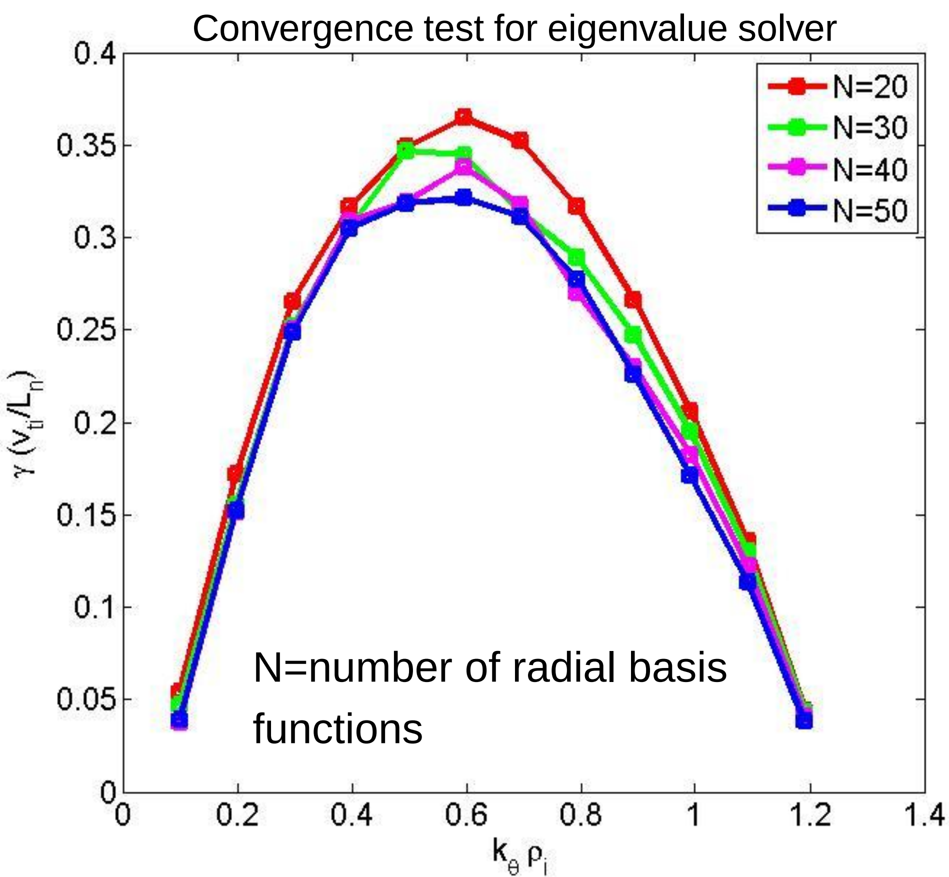
where

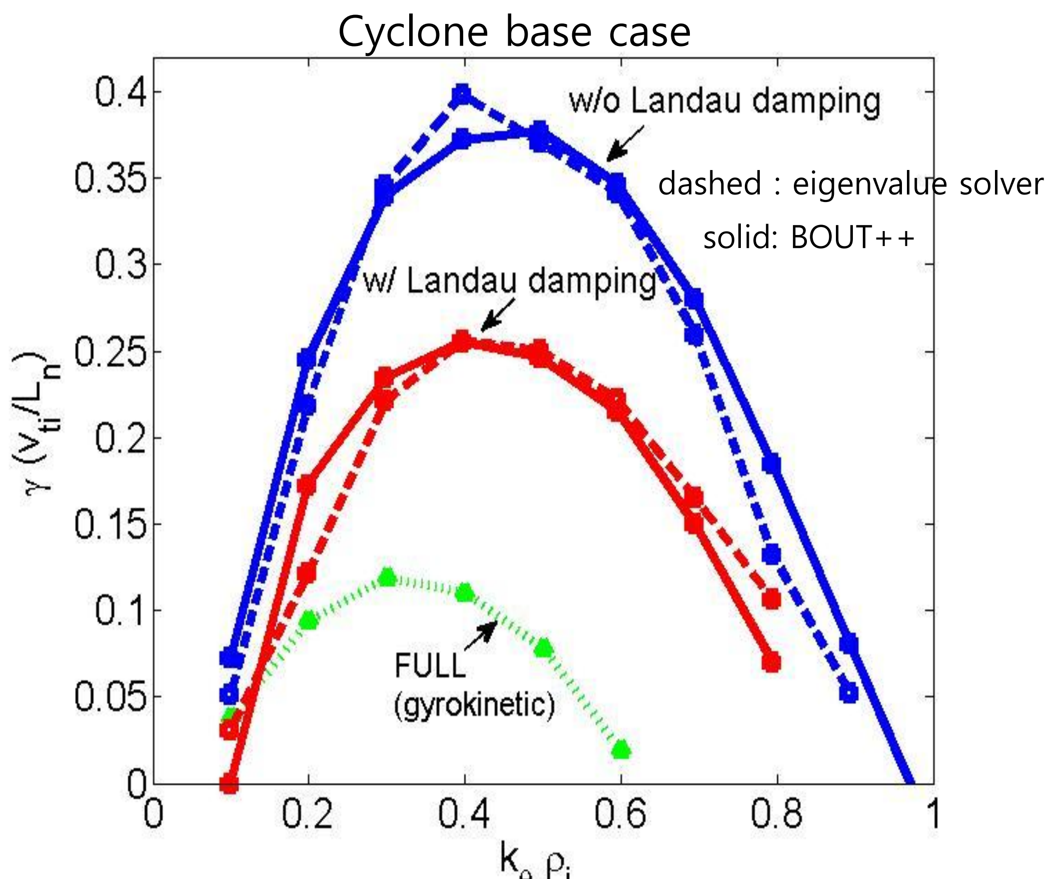
$$a_W^\pm(k,p') = \rho_*\frac{a}{R}\left[\int_0^1 \hat{n}_i\left\{(m\pm1)(1+\tau)\pm\left(\frac{\partial\ln\hat{n}_i}{\partial\rho} + \frac{\partial\ln\tau}{\partial\rho}\right)\tau\right\}W_{mnp}W_{m\pm1,n,p'}d\rho \pm \int_0^1 \hat{n}_i(1+\tau)W_{mnp}\frac{\partial W_{m\pm1,n,p'}}{\partial\rho}\rho d\rho\right]$$

$$b_W^\pm(k,p') = \rho_*\frac{a}{R}\left[\int_0^1\left(m\pm1+\frac{\partial\ln\hat{n}_i}{\partial\rho}\right)\hat{n}_iW_{mnp}W_{m\pm1,n,p'}d\rho \pm \int_0^1 \hat{n}_iW_{mnp}\frac{\partial W_{m\pm1,n,p'}}{\partial\rho}\rho d\rho\right], \qquad \tau = \frac{\hat{T}_i}{\hat{T}_e}$$

Comparison of BOUT++ with eigenvalue solver

- In eigenvalue solver, $W_{mnp} = \frac{\sqrt{2}}{J_{m+1}(\alpha_{mp})} J_m\left(\frac{\alpha_{mp}}{a} r\right) e^{i(m\theta - n\zeta)}\ (p = 1 \sim N)$ is used as basis function.
- The fluid equations are then projected on to the set of basis functions.
- Eigenvalues are obtained by using matlab.
- BOUT++ results agree well with eigenvalues, which means BOUT++ correctly solves the given equations.





Gyrofluid Simulation using BOUT++ GLF Code in Core Region

Gyrofluid equation for ITG

(M. A. Beer and G. W. Hammett, PoP '96)

- **Guiding center density equation**

$$\frac{\partial \tilde{n}_i}{\partial t}+\mathbf{V}_{\tilde{\Phi}}\cdot\nabla\tilde{n}_i+\frac{n_0}{2T_{i0}}\left[\hat{\nabla}_\perp^2\mathbf{V}_{\tilde{\Phi}}\right]\cdot\nabla\tilde{T}_{i\perp}+B\nabla_\parallel\frac{n_0\tilde{V}_{i\parallel}}{B}-\frac{en_0}{T_{i0}}\left(1+\frac{1}{2}\eta_i\hat{\nabla}_\perp^2\right)i\omega_*\tilde{\Phi}+\frac{en_0}{T_{i0}}\left(2+\frac{1}{2}\hat{\nabla}_\perp^2\right)i\omega_d\tilde{\Phi}+\frac{1}{T_{i0}}i\omega_d\left(\tilde{p}_{i\parallel}+\tilde{p}_{i\perp}\right)=0$$

- **Guiding center parallel velocity equation**

$$\frac{\partial \tilde{V}_{i\parallel}}{\partial t}+\mathbf{V}_{\tilde{\Phi}}\cdot\nabla\tilde{V}_{i\parallel}+\frac{e}{m_i}\nabla_\parallel\tilde{\Phi}+\frac{B}{m_in_0}\nabla_\parallel\frac{\tilde{p}_{i\parallel}}{B}+\left(\frac{\tilde{T}_{i\perp}}{m_i}+\frac{e}{2m_i}\hat{\nabla}_\perp^2\tilde{\Phi}\right)\nabla_\parallel\ln B+\frac{1}{p_{i0}}i\omega_d\left(\tilde{q}_{i\parallel}+\tilde{q}_{i\perp}\right)+4i\omega_d\tilde{V}_{i\parallel}=0$$

- **Guiding center parallel & perpendicular pressure equation**

$$\frac{\partial \tilde{p}_{i\parallel}}{\partial t}+\mathbf{V}_{\tilde{\Phi}}\cdot\nabla\tilde{p}_{i\parallel}+\frac{n_0}{2}\left[\hat{\nabla}_\perp^2\mathbf{V}_{\tilde{\Phi}}\right]\cdot\nabla\tilde{T}_{i\perp}+B\nabla_\parallel\frac{1}{B}\left(\tilde{q}_{i\parallel}+p_{i0}\tilde{V}_{i\parallel}\right)+2p_{i0}B\nabla_\parallel\frac{\tilde{V}_{i\parallel}}{B}+2\tilde{q}_{i\perp}\nabla_\parallel\ln B$$

$$-en_0\left(1+\eta_i+\frac{1}{2}\eta_i\hat{\nabla}_\perp^2\right)i\omega_*\tilde{\Phi}+en_0\left(4+\frac{1}{2}\hat{\nabla}_\perp^2\right)i\omega_d\tilde{\Phi}+4i\omega_d\tilde{p}_{i\parallel}+n_0\,i\omega_d\left(3\tilde{T}_{i\parallel}+\tilde{T}_{i\perp}\right)+2n_0\left|\omega_d\right|\left(\nu_1\tilde{T}_{i\parallel}+\nu_2\tilde{T}_{i\perp}\right)=0$$

$$\frac{\partial \tilde{p}_{i\perp}}{\partial t}+\mathbf{V}_{\tilde{\Phi}}\cdot\nabla\tilde{p}_{i\perp}+\frac{1}{2}\left[\hat{\nabla}_\perp^2\mathbf{V}_{\tilde{\Phi}}\right]\cdot\nabla\tilde{p}_{i\perp}+n_0\left[\hat{\hat{\nabla}}_\perp^2\mathbf{V}_{\tilde{\Phi}}\right]\cdot\nabla\tilde{T}_{i\perp}+B^2\nabla_\parallel\frac{1}{B^2}\left(\tilde{q}_{i\perp}+p_{i0}\tilde{V}_{i\parallel}\right)$$

$$-en_0\left[1+\frac{1}{2}\hat{\nabla}_\perp^2+\eta_i\left(1+\frac{1}{2}\hat{\nabla}_\perp^2+\hat{\hat{\nabla}}_\perp^2\right)\right]i\omega_*\tilde{\Phi}+en_0\left(3+\frac{3}{2}\hat{\nabla}_\perp^2+\hat{\hat{\nabla}}_\perp^2\right)i\omega_d\tilde{\Phi}+3i\omega_d\tilde{p}_{i\perp}+n_0i\omega_d\left(\tilde{T}_{i\parallel}+2\tilde{T}_{i\perp}\right)+2n_0\left|\omega_d\right|\left(\nu_3\tilde{T}_{i\parallel}+\nu_4\tilde{T}_{i\perp}\right)=0$$

- **Quasi-neutrality**

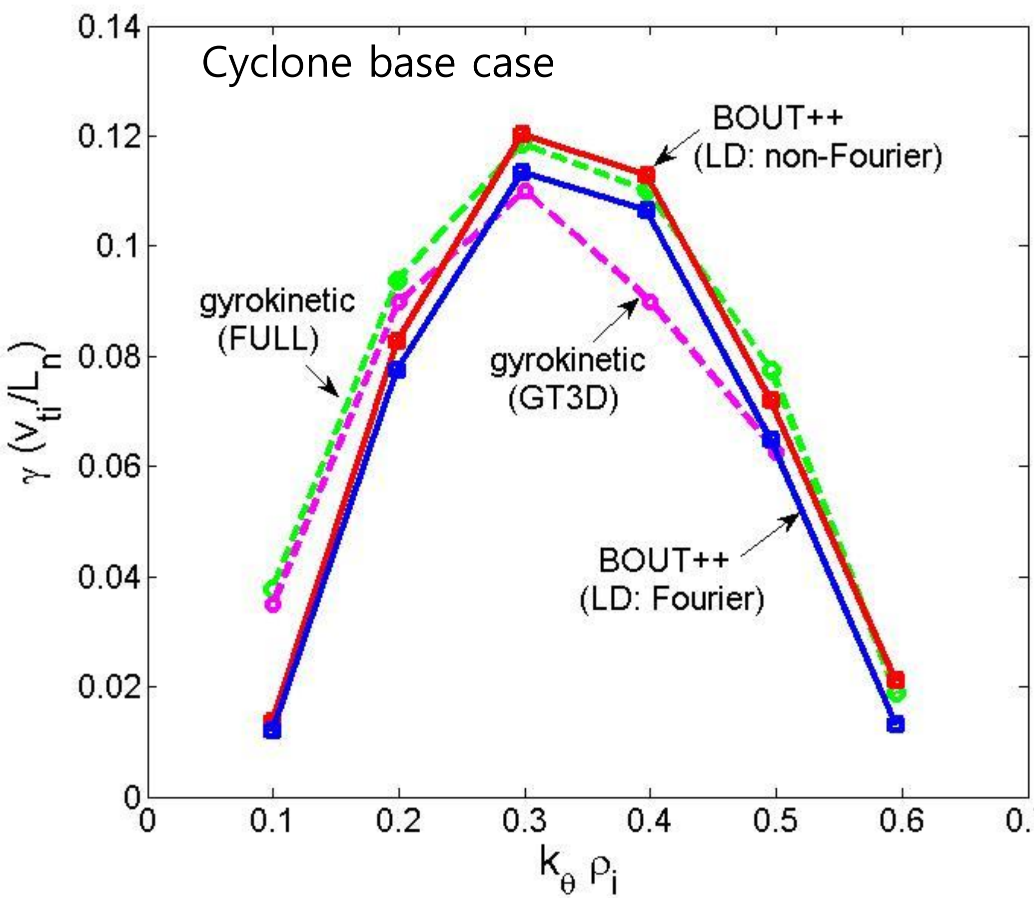
$$\frac{n_0}{T_{e0}}\tilde{\varphi}+\frac{n_0}{T_{i0}}\left(1-\Gamma_0\right)\tilde{\varphi}=\Gamma_0^{1/2}\tilde{n}_i+\frac{n_0}{T_{i0}}b\frac{\partial\Gamma_0^{1/2}}{\partial b}\tilde{T}_i$$

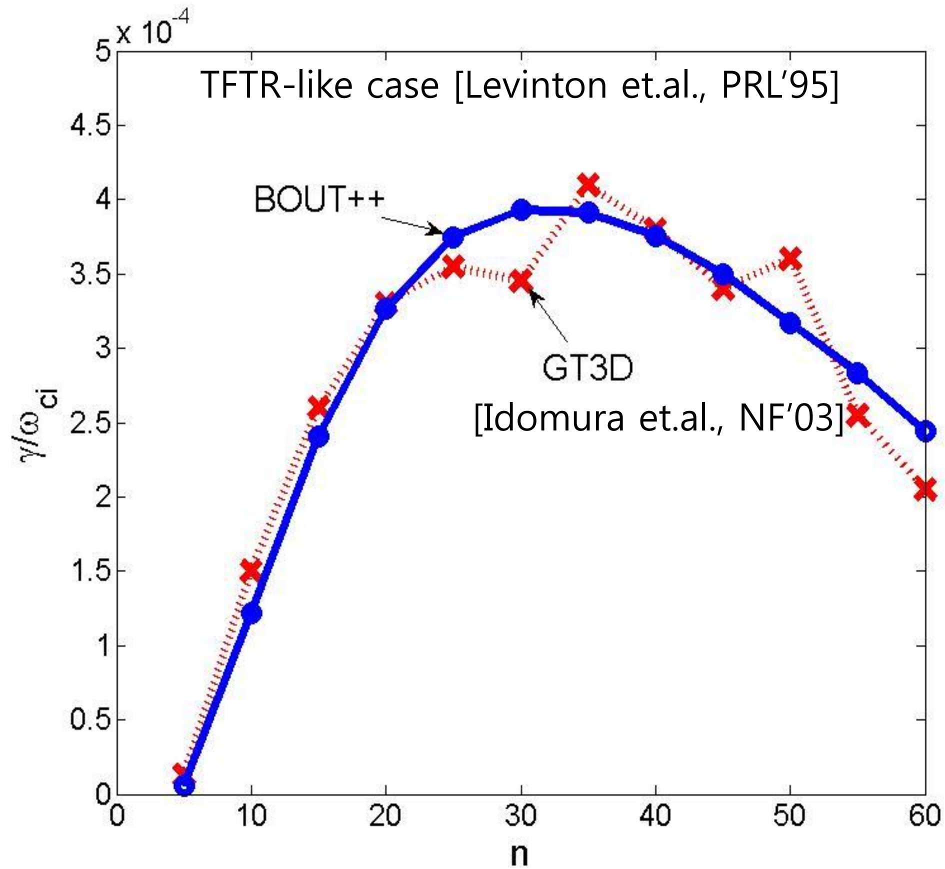
where $\tilde{\Phi}=\Gamma_0^{1/2}\tilde{\varphi},\quad \frac{1}{2}\hat{\nabla}_\perp^2\tilde{\Phi}=\left(b\frac{\partial\Gamma_0^{1/2}}{\partial b}\right)\tilde{\varphi},\quad \hat{\hat{\nabla}}_\perp^2\tilde{\Phi}=b\frac{\partial^2}{\partial b^2}\left(b\Gamma_0^{1/2}\right)\tilde{\varphi},\quad \Gamma_0^{1/2}\cong\frac{1}{1+b/2}$

$$i\omega_d=\left(v_t^2/\omega_c\right)\mathbf{b}\times\nabla\ln B\cdot\nabla,\quad b=-\rho_i^2\nabla_\perp^2$$

Linear benchmarks

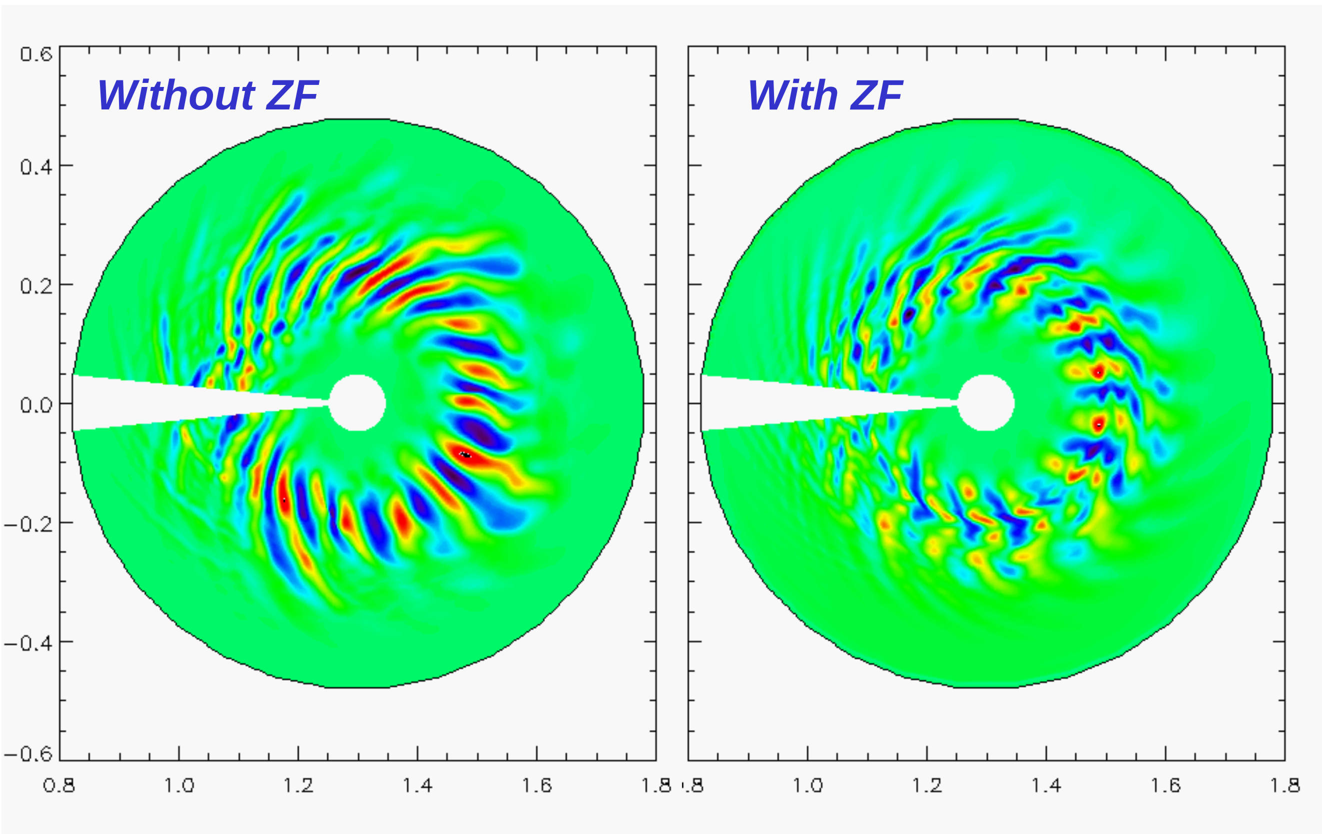
- BOUT++ using Beer's "3+1" field model agrees well with gyrokinetic codes.
- Non-Fourier method for Landau damping shows good agreement with Fourier method.



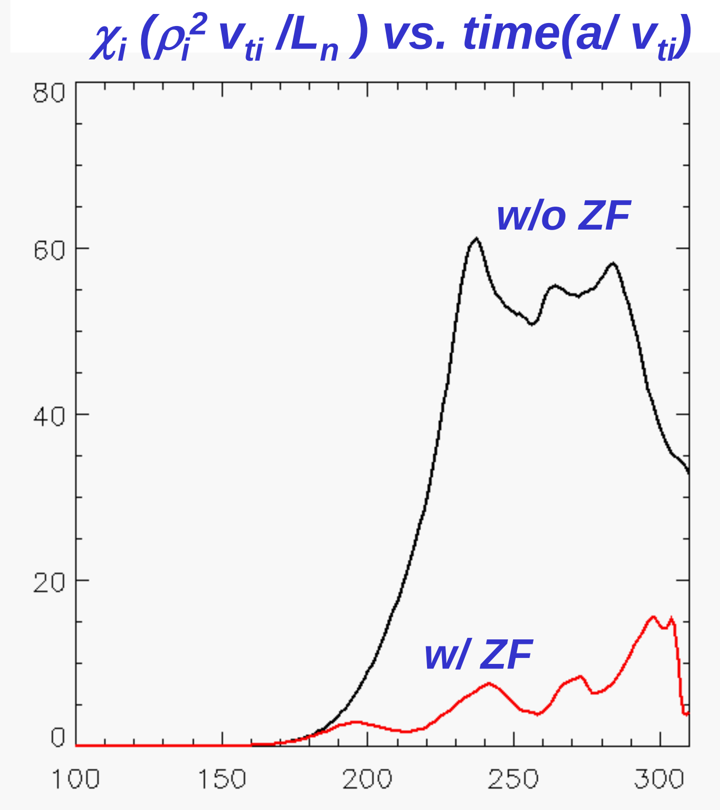


Nonlinear simulations of ITG

- Global nonlinear simulations using Beer model performed at fixed profile
- Turbulence suppression by zonal flow observed



Potential fluctuation



Hands-on Exercise for Eigenvalue Solver using MATLAB

0) Run matlab :

> module load matlab-nofonts
> matlab

1) Run eigensolver_init.m and eigensolver_ITG.m in matlab:

>> eigensolver_init : assign equilibrium profiles, calculate radial basis functions, arrange mode index, evaluate matrix elements involving integrals

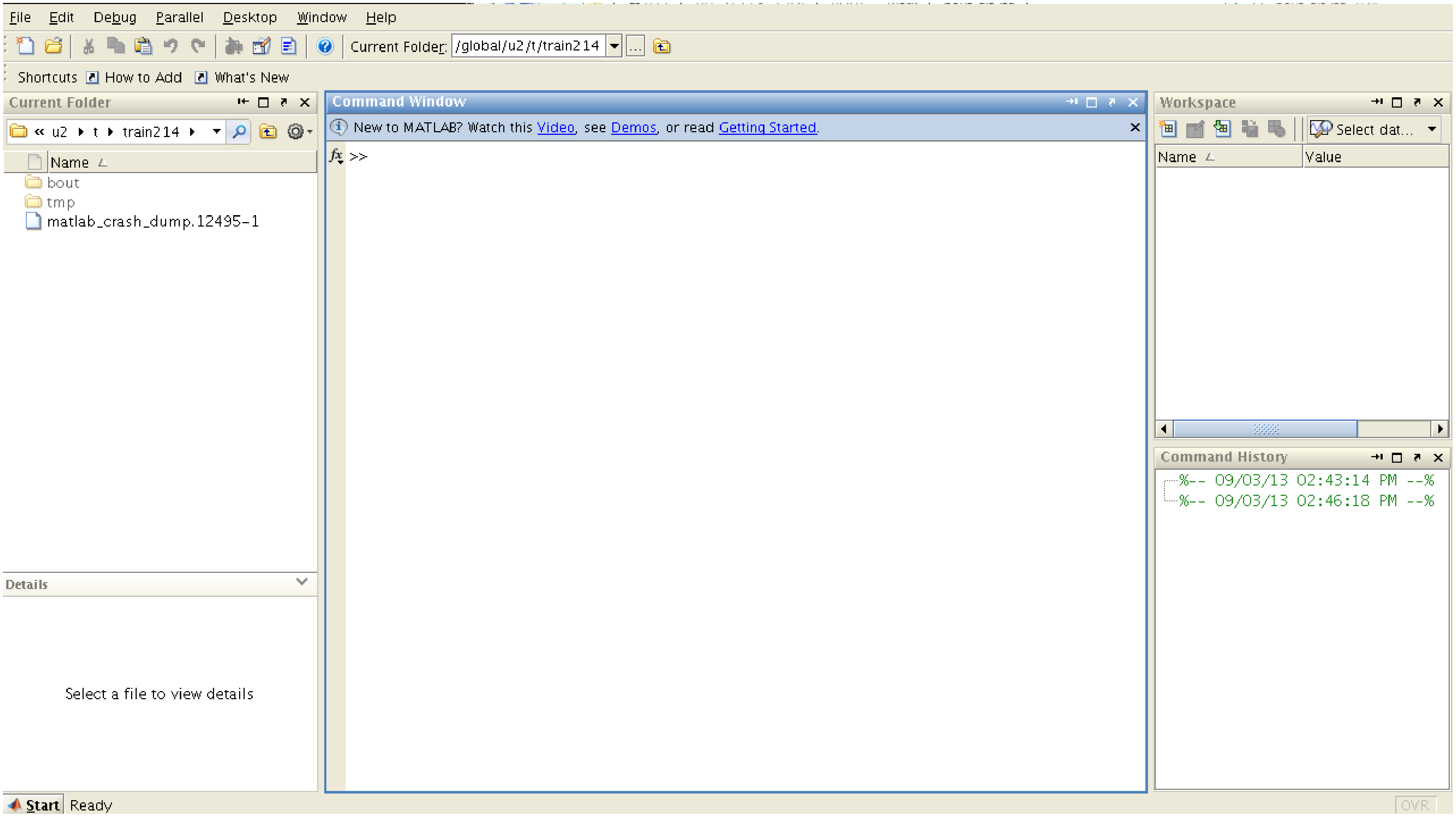
>> eigensolver_ITG : set up matrix and solve the eigenvalue equation for ITG

2) Run plot_eigenvalue.m and plot_eigenmode.m in matlab:

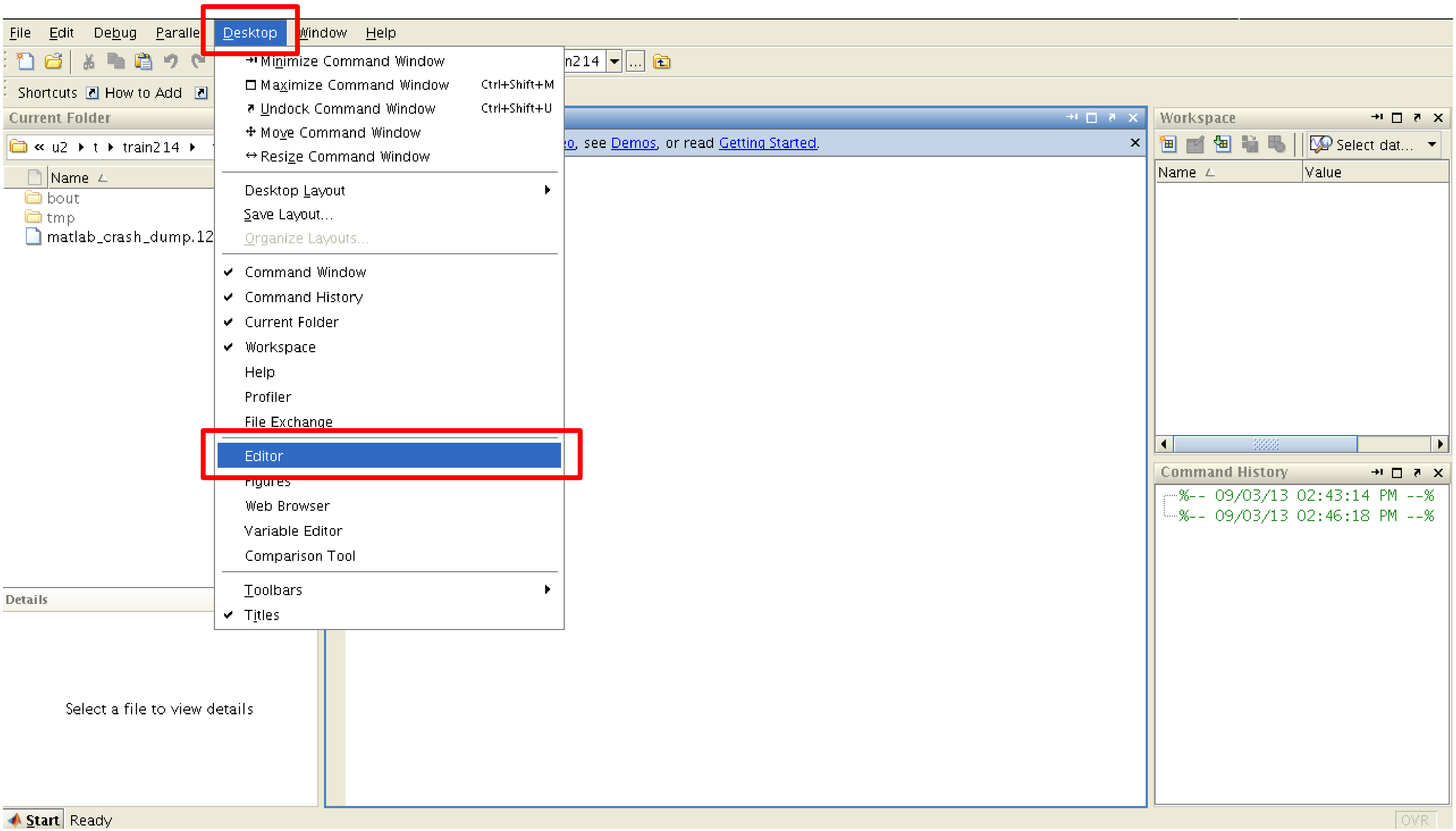
>> plot_eigenvalue : plot linear ITG growth rate and real frequency vs. k_theta*rho_i
>> plot_eigenmode : choose toroidal mode number and plot eigenmode structure

Run matlab : **> module load matlab-nofonts**

> matlab

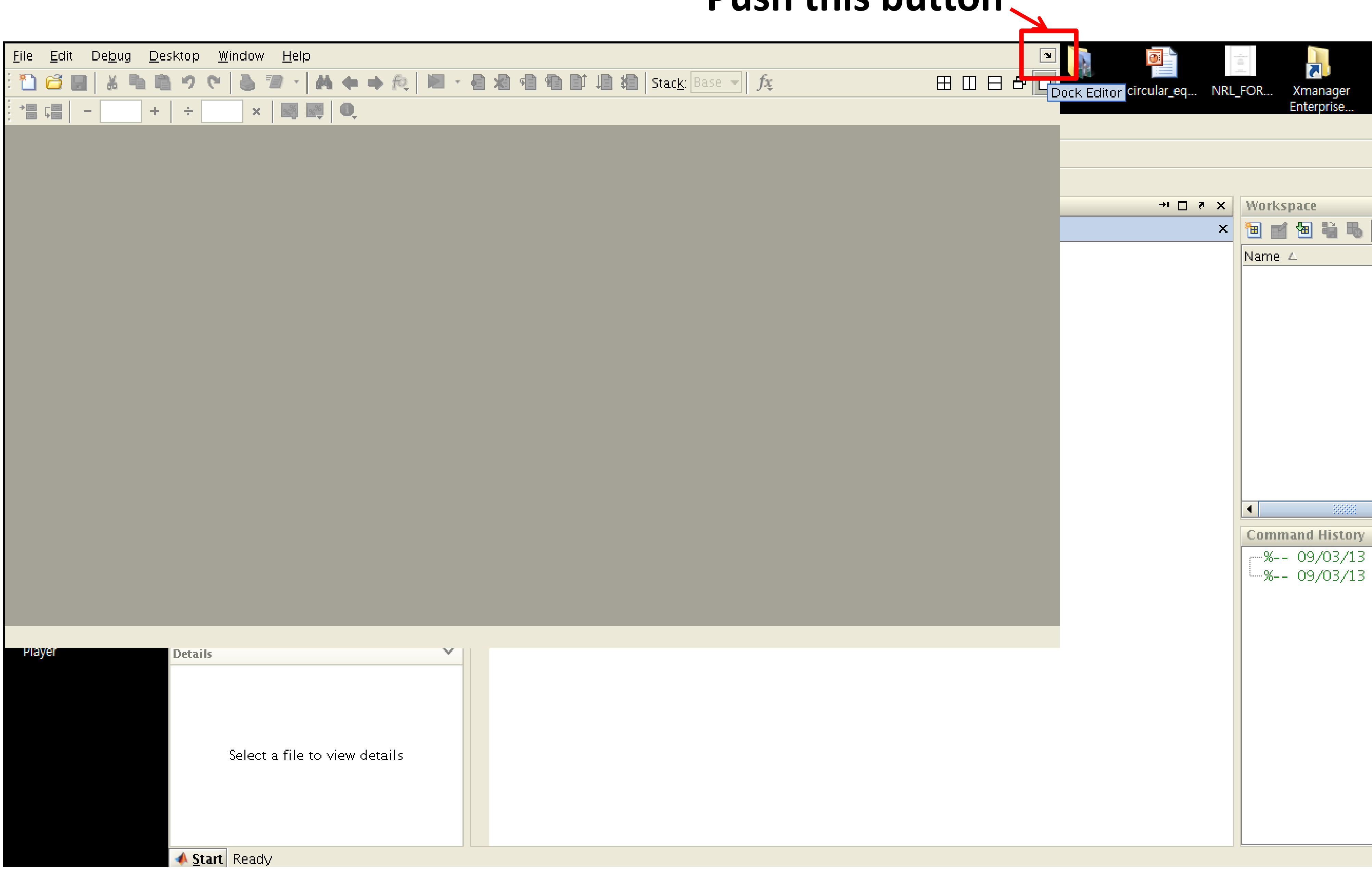


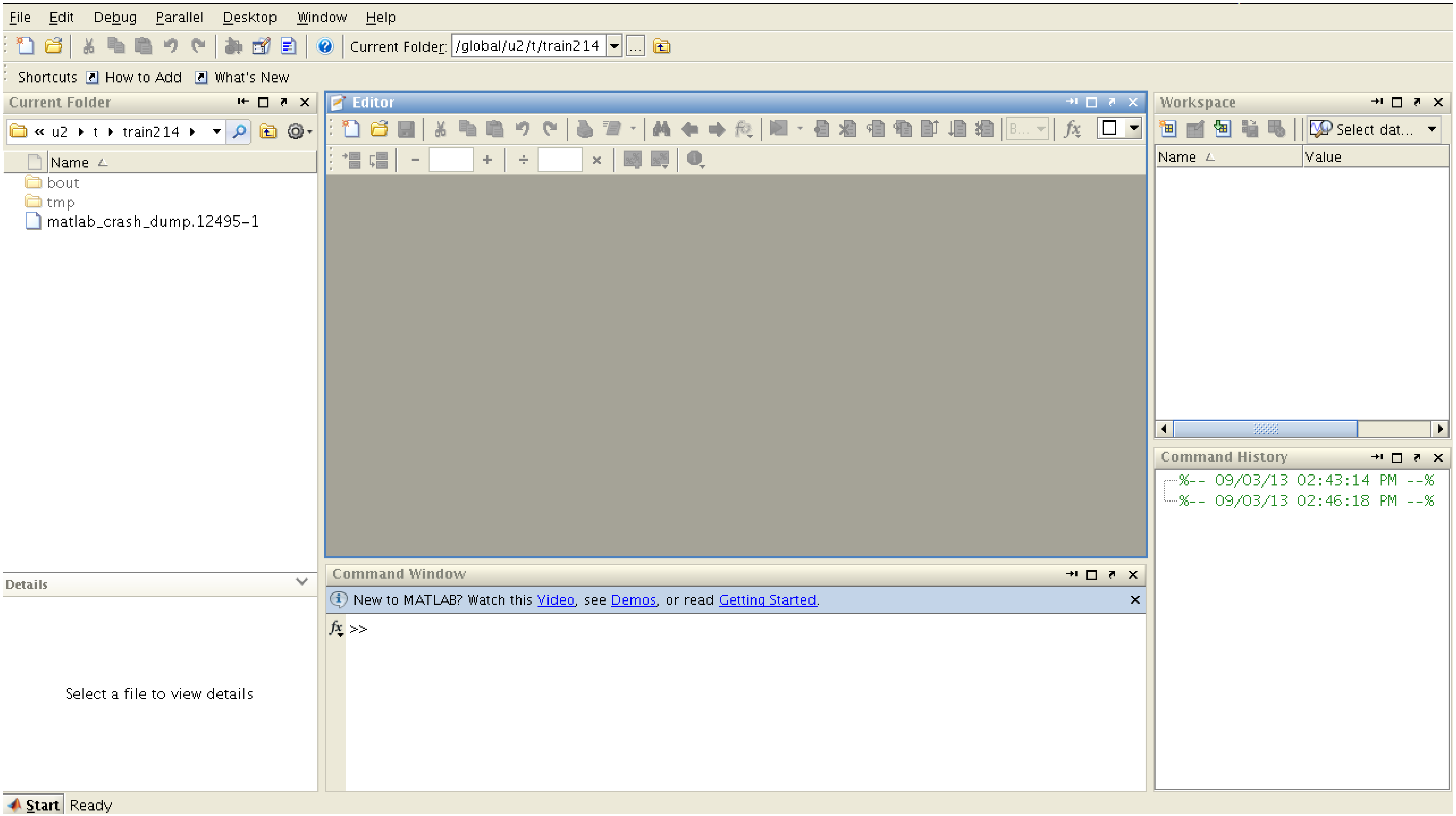
Run Editor :


File Edit Debug Paralle Desktop Window Help
Minimize Command Window
Maximize Command Window Ctrl+Shift+M
Undock Command Window Ctrl+Shift+U
Move Command Window
Resize Command Window
Desktop Layout
Save Layout...
Organize Layouts...
Command Window
Command History
Current Folder
Workspace
Help
Profiler
File Exchange
Editor
Figures
Web Browser
Variable Editor
Comparison Tool
Toolbars
Titles
Shortcuts How to Add
Current Folder
« u2 ▸ t ▸ train214
Name
bout
tmp
matlab_crash_dump.12
Details
Select a file to view details
see Demos, or read Getting Started.
Workspace
Select dat...
Name
Value
Command History
%-- 09/03/13 02:43:14 PM --%
%-- 09/03/13 02:46:18 PM --%
Start Ready
OVR

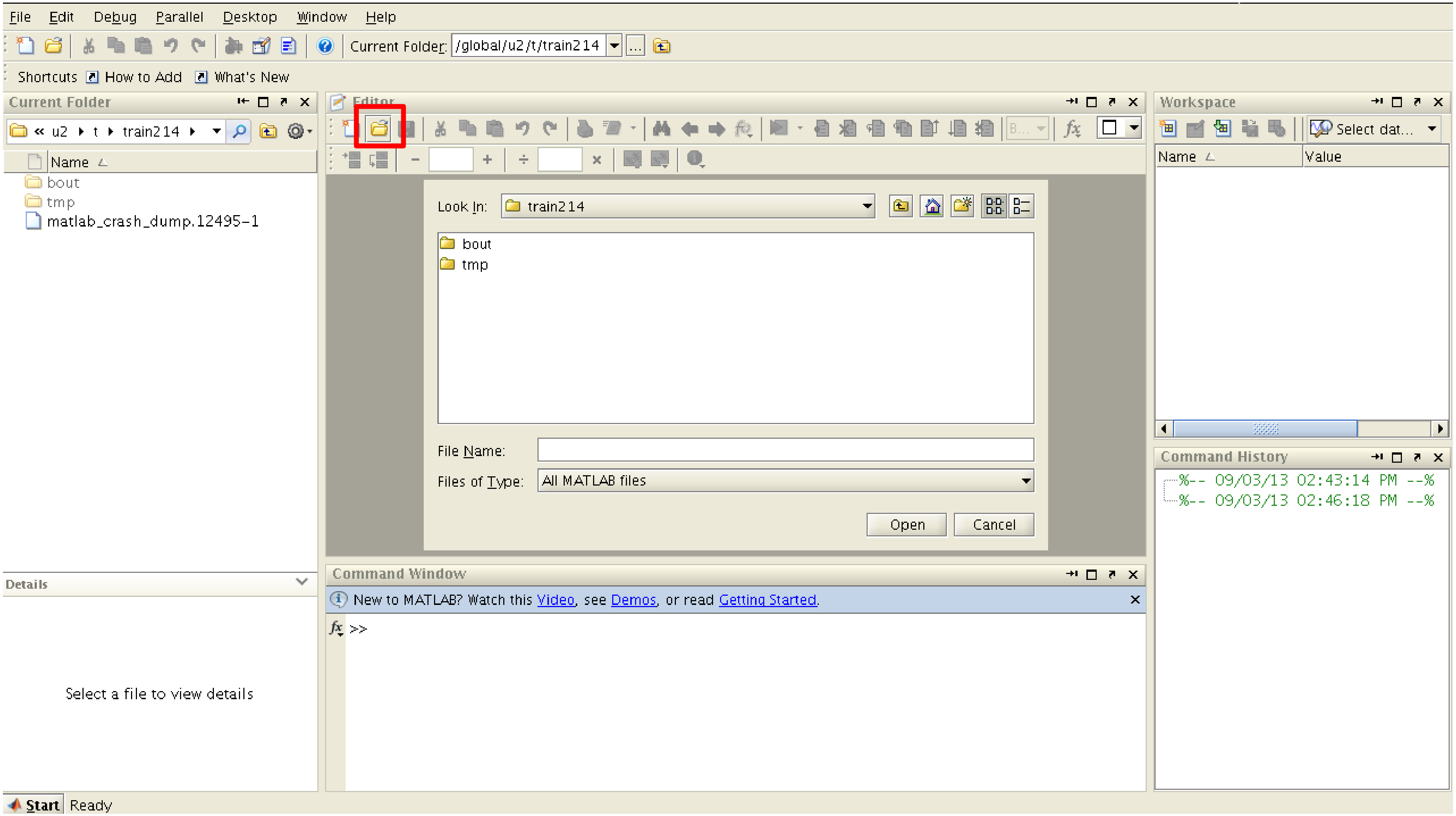
Dock Editor:

Push this button

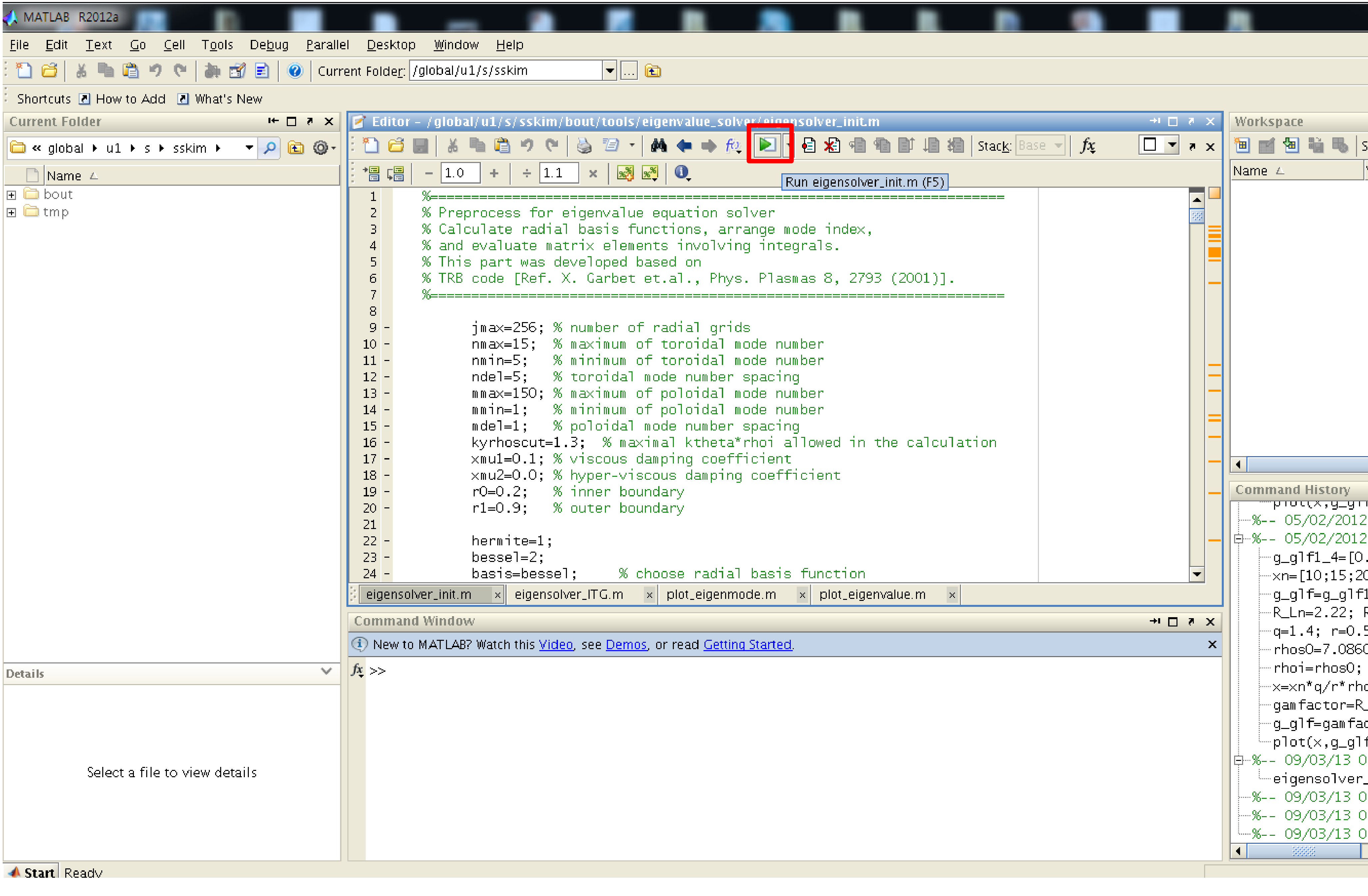


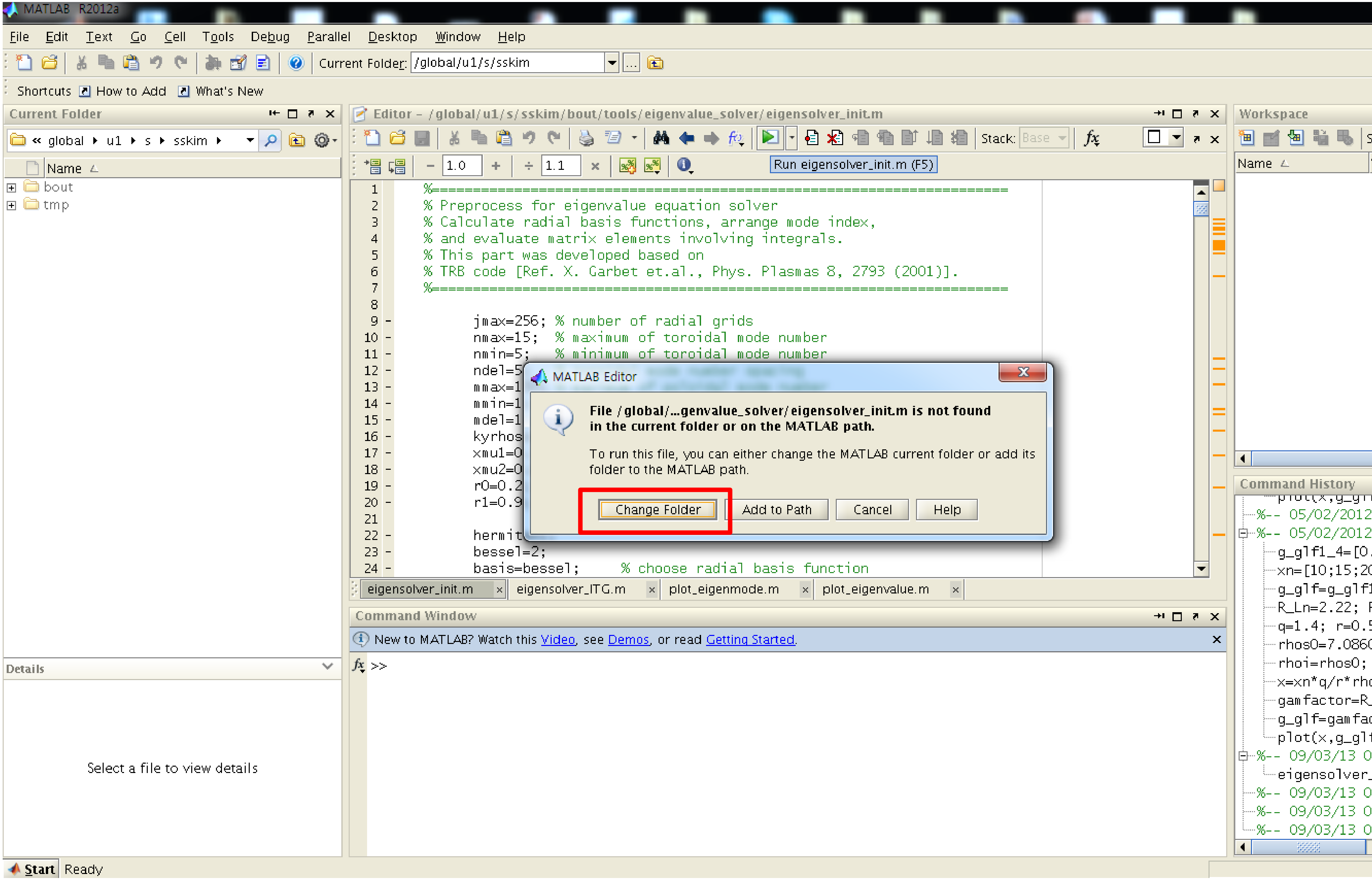

File Edit Debug Parallel Desktop Window Help
Current Folder: /global/u2/t/train214
Shortcuts How to Add What's New
Current Folder
« u2 t train214
Name
bout
tmp
matlab_crash_dump.12495-1
Details
Select a file to view details
Editor
Workspace
Select dat...
Name
Value
Command History
%-- 09/03/13 02:43:14 PM --%
%-- 09/03/13 02:46:18 PM --%
Command Window
New to MATLAB? Watch this Video, see Demos, or read Getting Started.
fx >>
Start Ready

Open eigensolver_init.m, eigensolver_ITG.m, plot_eigenvalue.m, plot_eigenmode.m in tools/eigenvalue_solver.



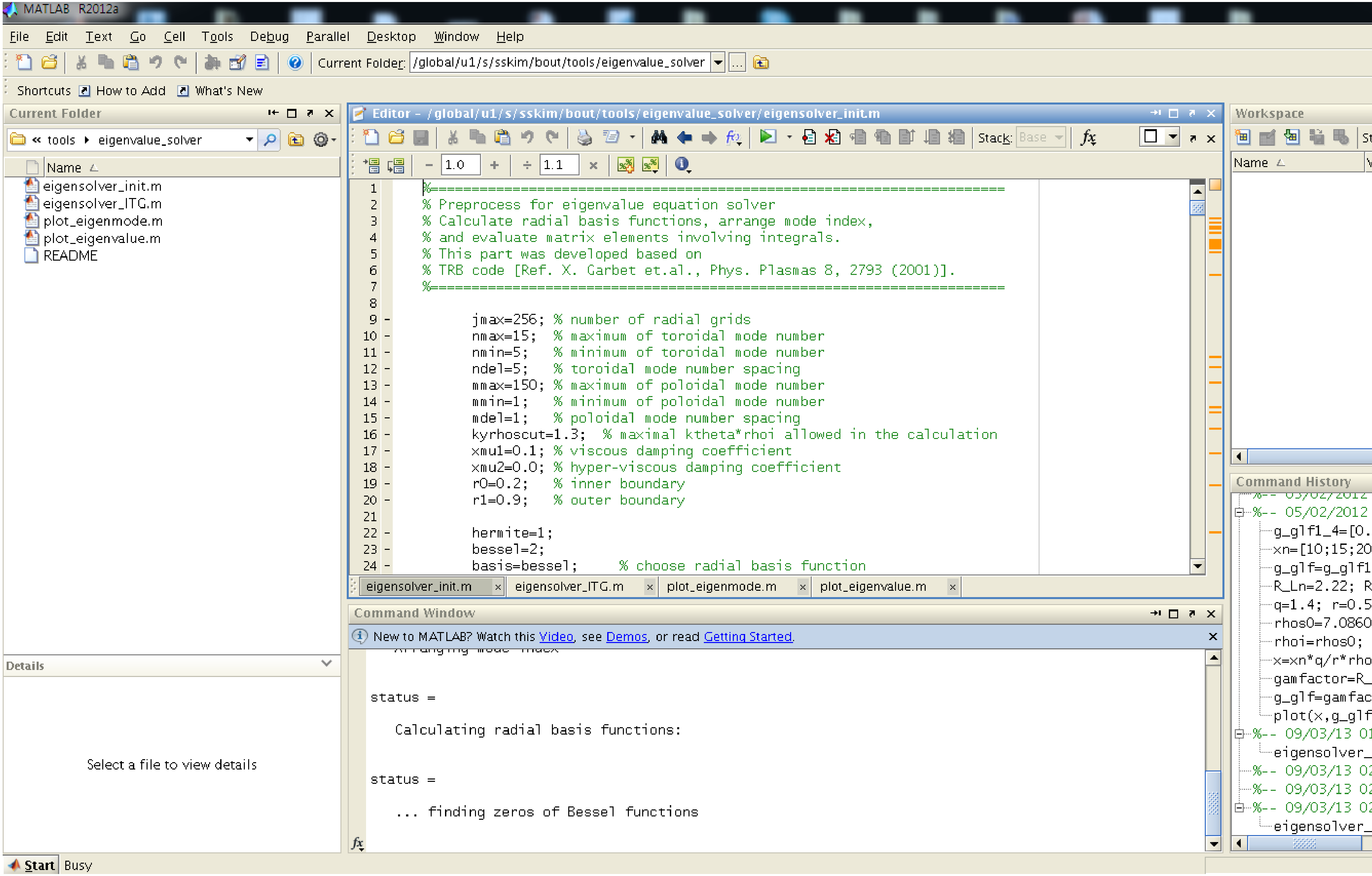
Run eigensolver_init.m;

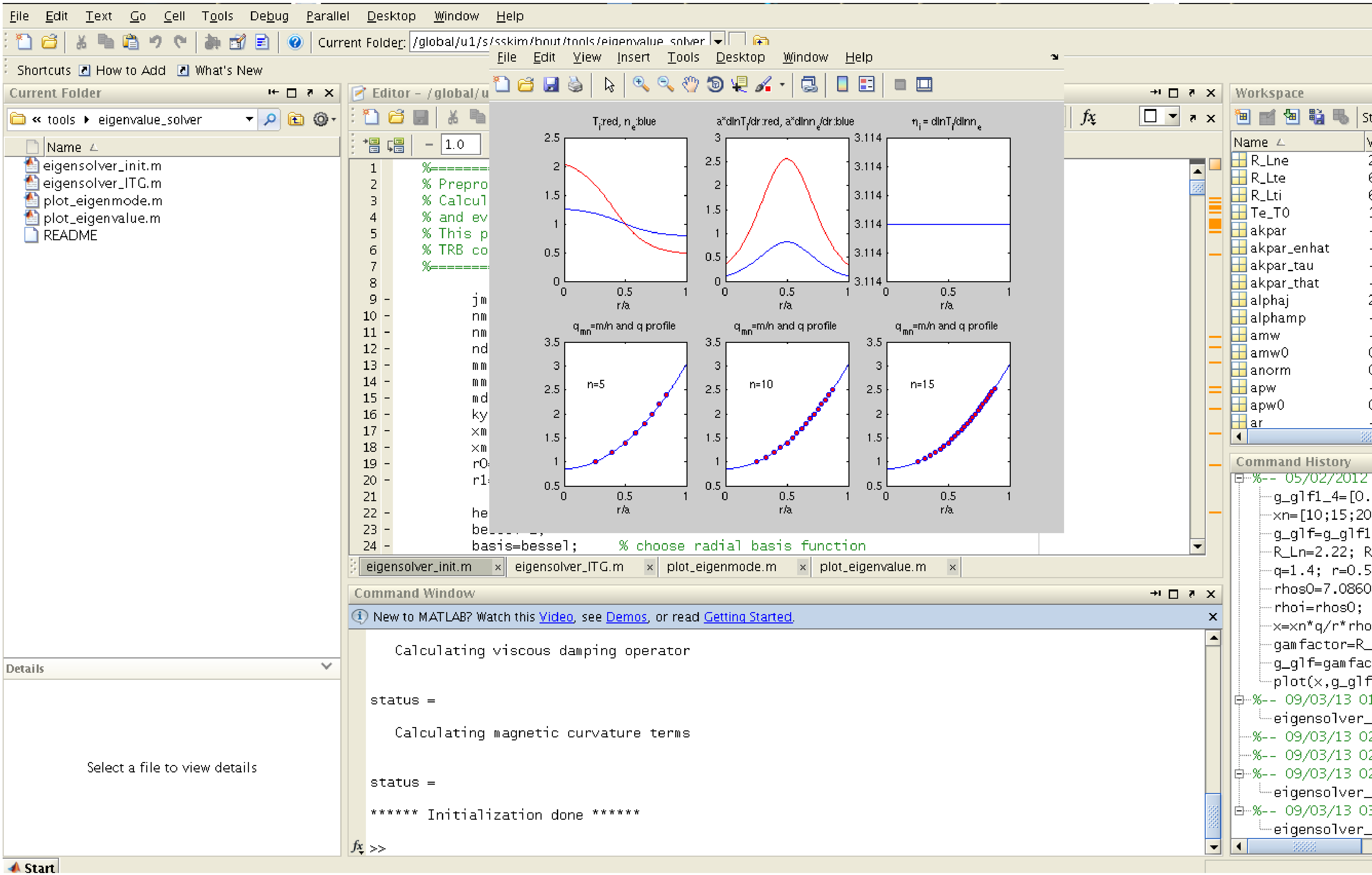




MATLAB R2012a
File Edit Text Go Cell Tools Debug Parallel Desktop Window Help
Current Folder: /global/u1/s/sskim
Shortcuts How to Add What's New
Current Folder
« global ▸ u1 ▸ s ▸ sskim ▸
Name
bout
tmp
Details
Select a file to view details
Editor – /global/u1/s/sskim/bout/tools/eigenvalue_solver/eigensolver_init.m
Stack: Base
Run eigensolver_init.m (F5)
%===
% Preprocess for eigenvalue equation solver
% Calculate radial basis functions, arrange mode index,
% and evaluate matrix elements involving integrals.
% This part was developed based on
% TRB code [Ref. X. Garbet et.al., Phys. Plasmas 8, 2793 (2001)].
%===
jmax=256; % number of radial grids
nmax=15; % maximum of toroidal mode number
nmin=5; % minimum of toroidal mode number
bessel=2;
basis=bessel; % choose radial basis function
MATLAB Editor
File /global/...genvalue_solver/eigensolver_init.m is not found in the current folder or on the MATLAB path.
To run this file, you can either change the MATLAB current folder or add its folder to the MATLAB path.
Change Folder
Add to Path
Cancel
Help
eigensolver_init.m
eigensolver_ITG.m
plot_eigenmode.m
plot_eigenvalue.m
Command Window
New to MATLAB? Watch this Video, see Demos, or read Getting Started.
>>
Workspace
Name
Command History
Start
Ready

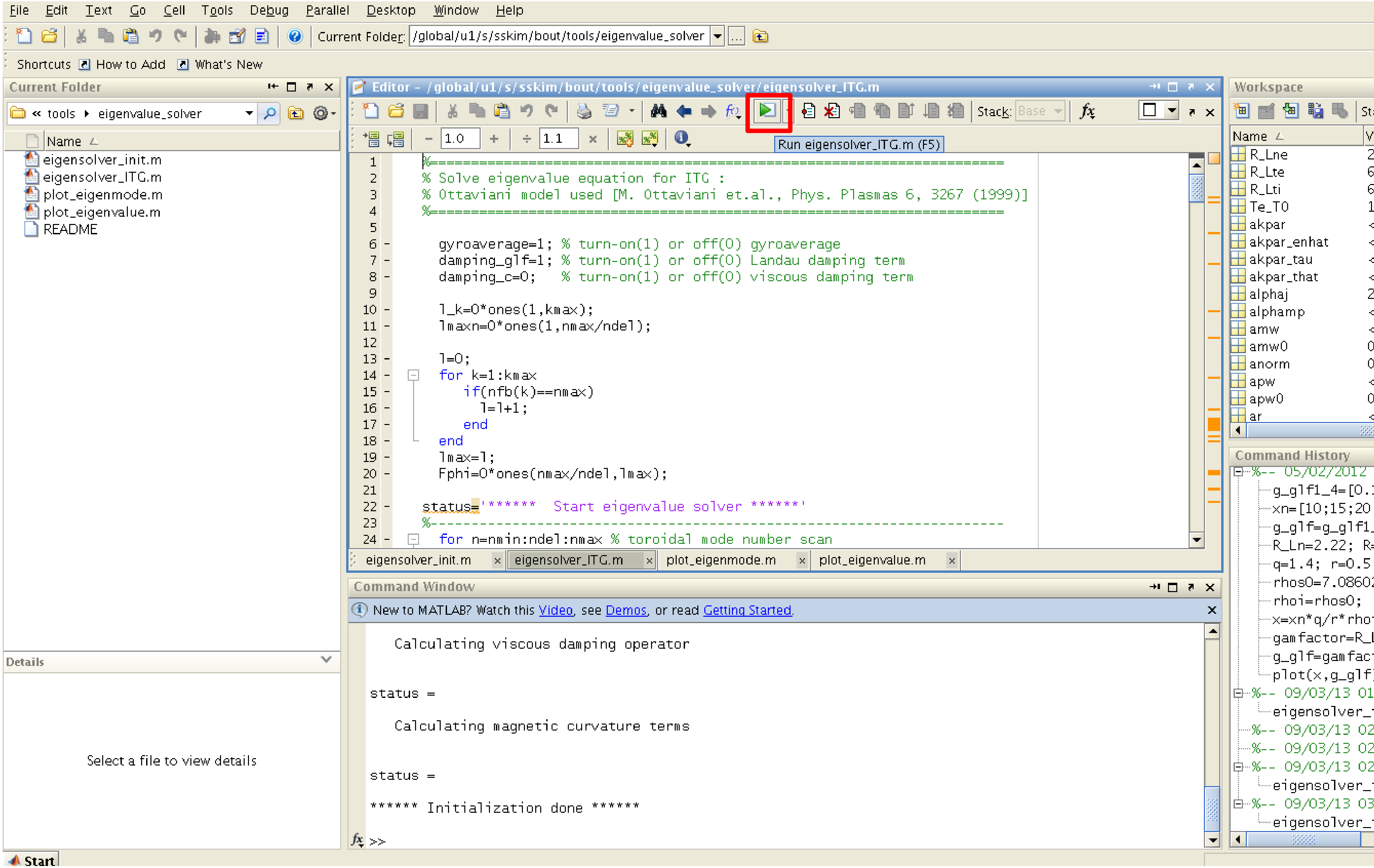
Running eigensolver_init.m … : it takes about 5 min.



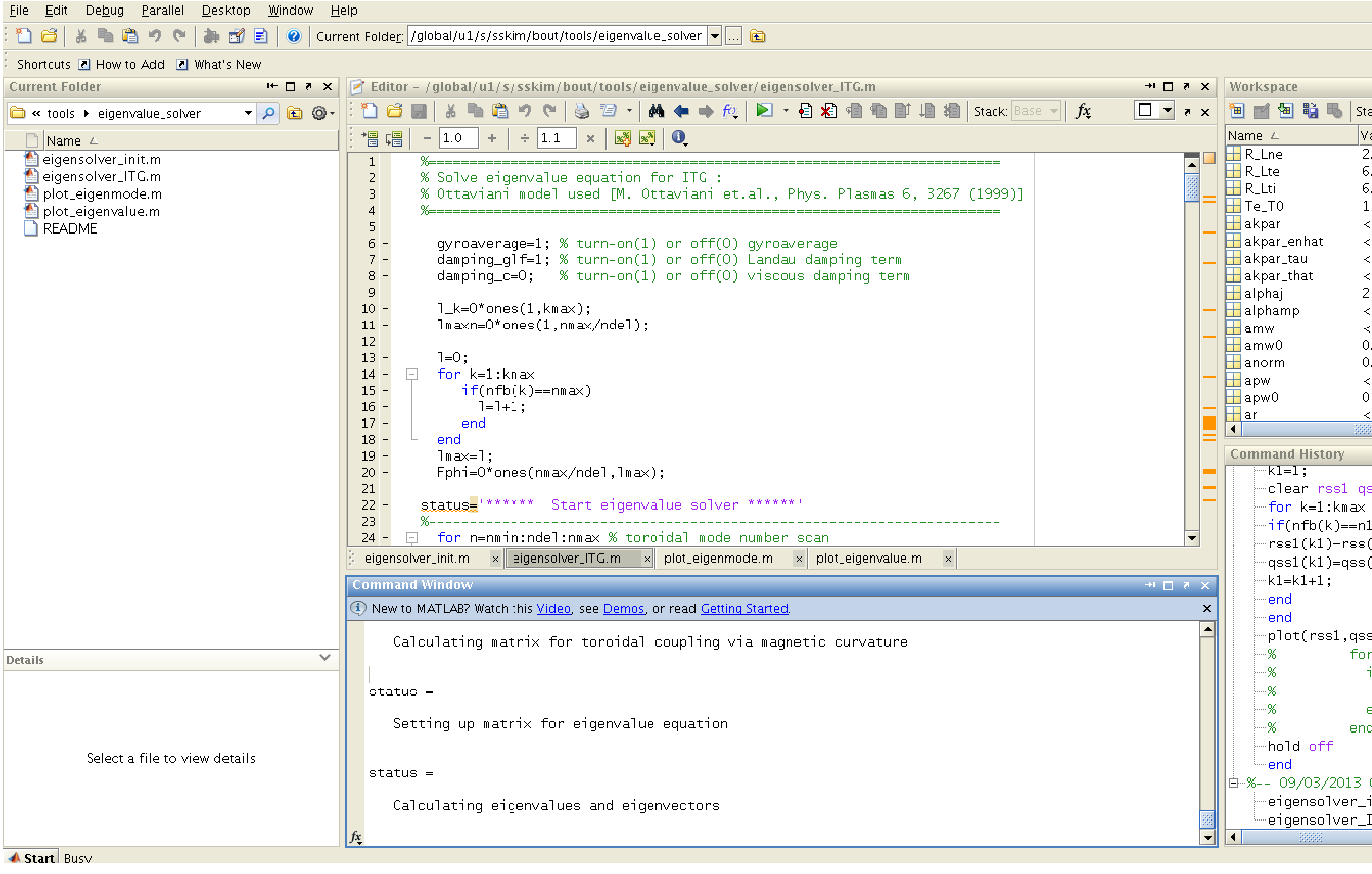


File Edit Text Go Cell Tools Debug Parallel Desktop Window Help
Current Folder: /global/u1/s/sskim/bout/tools/eigenvalue_solver
Shortcuts How to Add What's New
Current Folder
tools eigenvalue_solver
Name
eigensolver_init.m
eigensolver_ITG.m
plot_eigenmode.m
plot_eigenvalue.m
README
Details
Select a file to view details
Editor - /global/u
% Prepro
% Calcul
% and ev
% This p
% TRB co
basis=bessel; % choose radial basis function
eigensolver_init.m
eigensolver_ITG.m
plot_eigenmode.m
plot_eigenvalue.m
File Edit View Insert Tools Desktop Window Help
T_i:red, n_e:blue
a*dlnT_i/dr:red, a*dlnn_e/dr:blue
η_i = dlnT_i/dlnn_e
q_mn=m/n and q profile
n=5
n=10
n=15
r/a
Command Window
New to MATLAB? Watch this Video, see Demos, or read Getting Started.
Calculating viscous damping operator
status =
Calculating magnetic curvature terms
status =
****** Initialization done ******
>>
Workspace
Name
R_Lne
R_Lte
R_Lti
Te_T0
akpar
akpar_enhat
akpar_tau
akpar_that
alphaj
alphamp
amw
amw0
anorm
apw
apw0
ar
Command History
g_glf1_4=[0.1
xn=[10;15;20;
g_glf=g_glf1_
R_Ln=2.22; R=
q=1.4; r=0.5;
rhos0=7.08602
rhoi=rhos0;
x=xn*q/r*rhoi
gamfactor=R_L
g_glf=gamfact
plot(x,g_glf)
eigensolver_i
Start

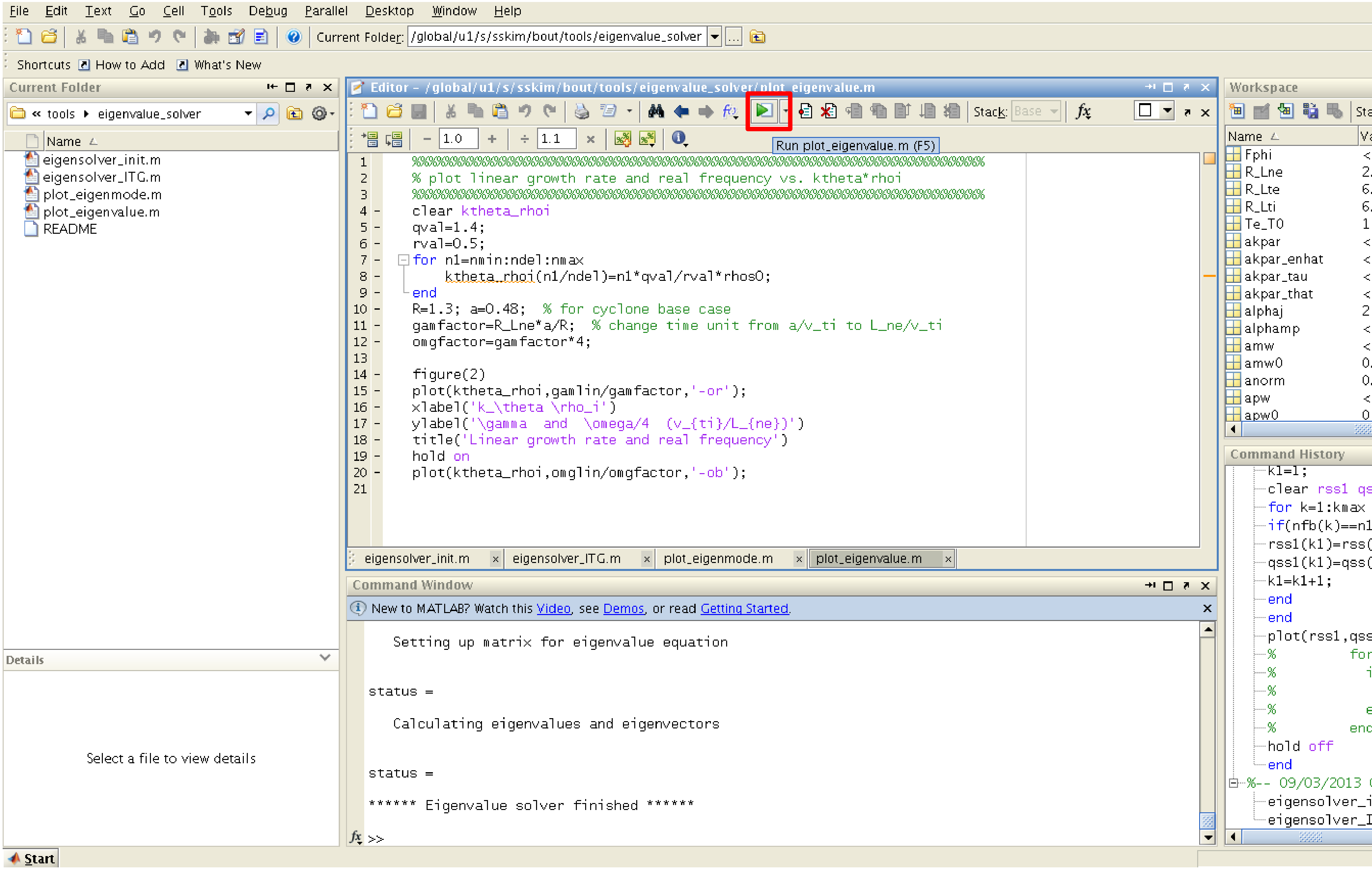
Run eigensolver_ITG.m;

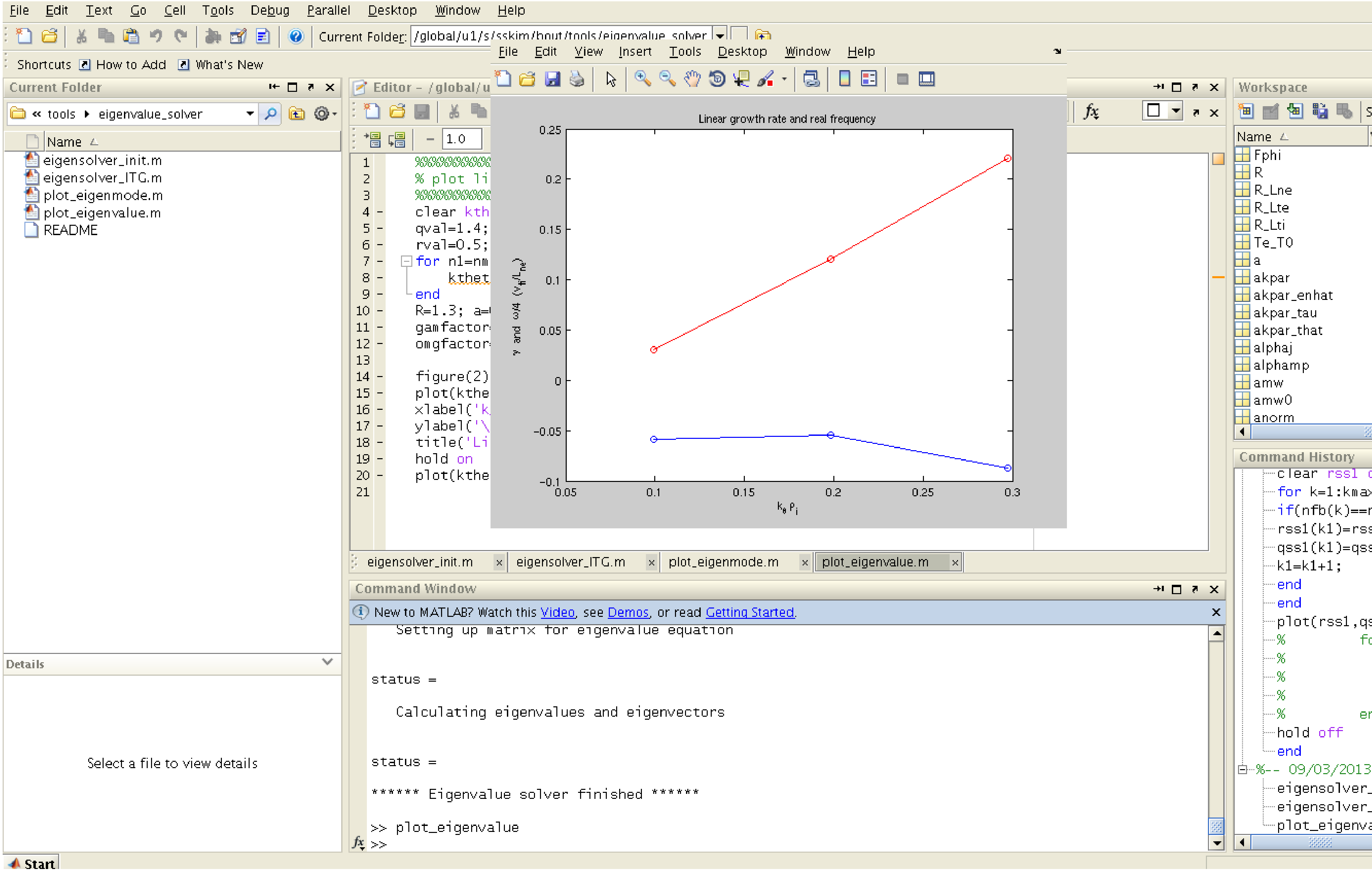


Running eigensolver_ITG.m … : it takes about 10 min.

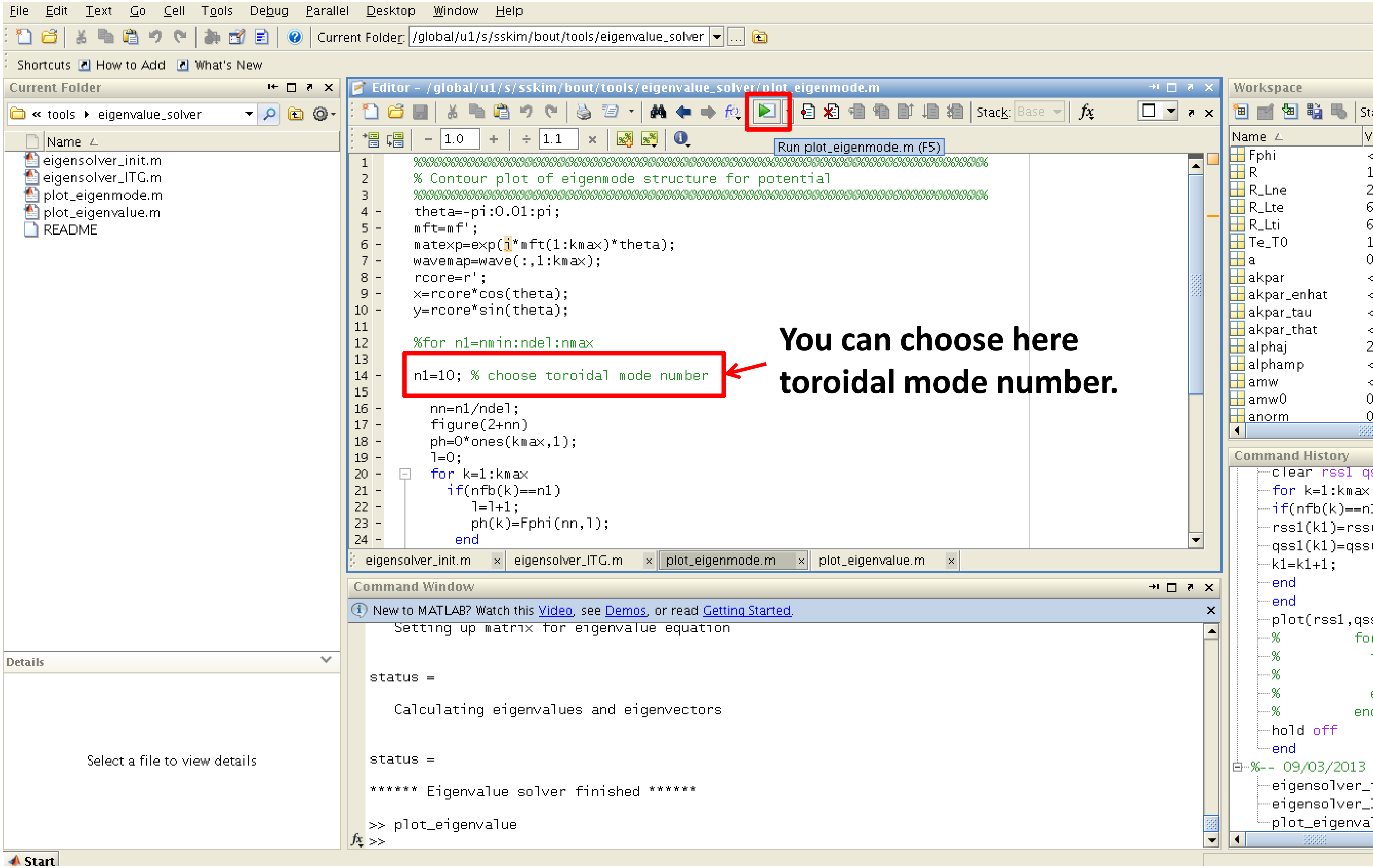


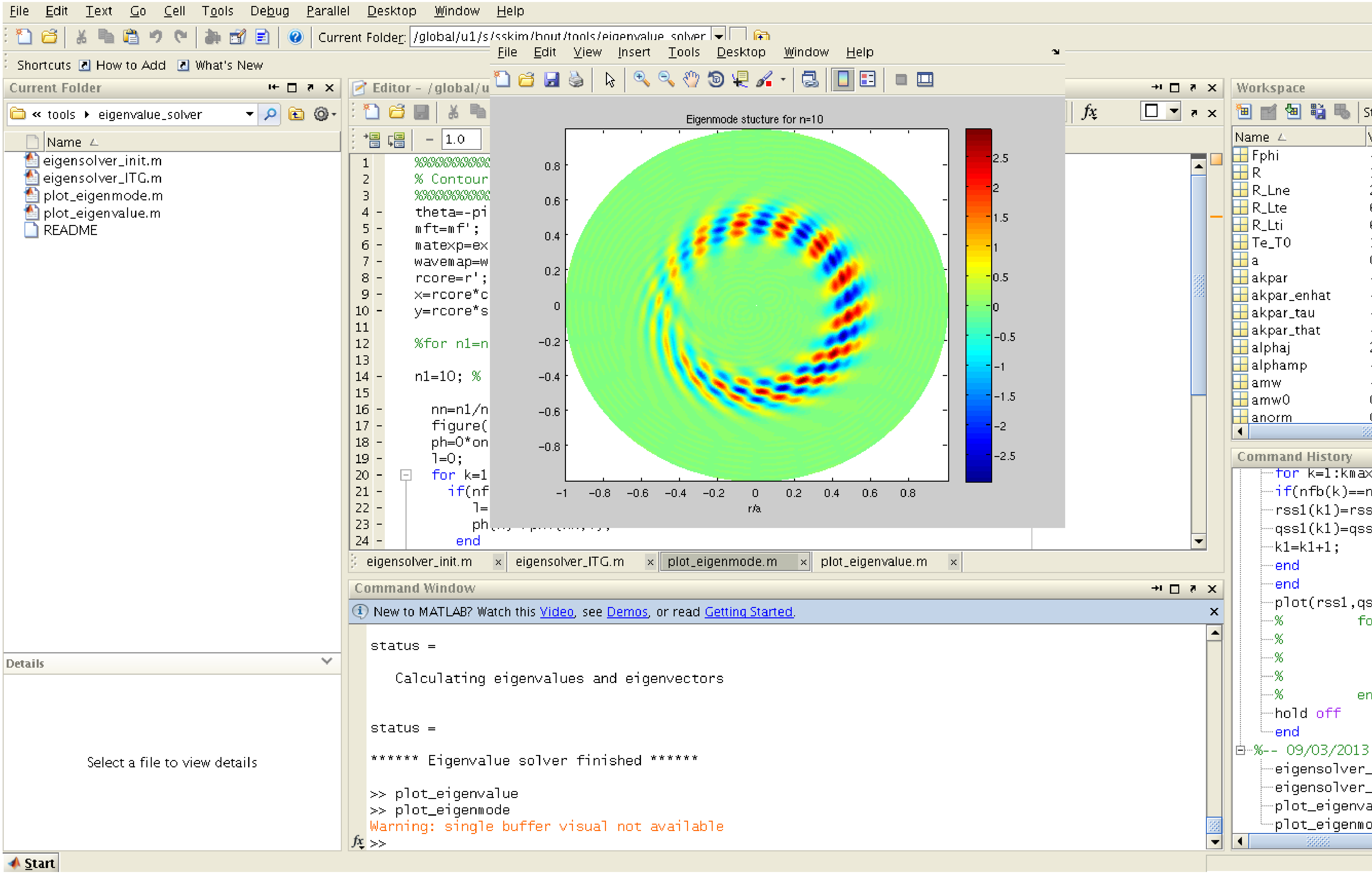
Run plot_eigenvalue.m;




File Edit Text Go Cell Tools Debug Parallel Desktop Window Help
Current Folder: /global/u1/s/sskim/bout/tools/eigenvalue_solver
Shortcuts How to Add What's New
Current Folder
tools eigenvalue_solver
Name
eigensolver_init.m
eigensolver_ITG.m
plot_eigenmode.m
plot_eigenvalue.m
README
Details
Select a file to view details
Editor – /global/u
%%%%%%%%%%
% plot li
clear kth
qval=1.4;
rval=0.5;
for n1=nm
kthet
end
R=1.3; a=
gamfactor
omgfactor
figure(2)
plot(kthe
xlabel('k
ylabel('\
title('Li
hold on
plot(kthe
eigensolver_init.m eigensolver_ITG.m plot_eigenmode.m plot_eigenvalue.m
File Edit View Insert Tools Desktop Window Help
Linear growth rate and real frequency
γ and ω/4 (v_{ti}/L_{ne})
$k_\theta \rho_i$
0.25 0.2 0.15 0.1 0.05 0 -0.05 -0.1
0.05 0.1 0.15 0.2 0.25 0.3
Command Window
New to MATLAB? Watch this Video, see Demos, or read Getting Started.
Setting up matrix for eigenvalue equation
status =
Calculating eigenvalues and eigenvectors
status =
****** Eigenvalue solver finished ******
>> plot_eigenvalue
>>
Workspace
Name
Fphi
R
R_Lne
R_Lte
R_Lti
Te_T0
a
akpar
akpar_enhat
akpar_tau
akpar_that
alphaj
alphamp
amw
amw0
anorm
Command History
clear rss1 qs
for k=1:kmax
if(nfb(k)==n1
rss1(k1)=rss(
qss1(k1)=qss(
k1=k1+1;
end
end
plot(rss1,qss
hold off
end
%-- 09/03/2013
eigensolver_i
eigensolver_I
plot_eigenval
Start

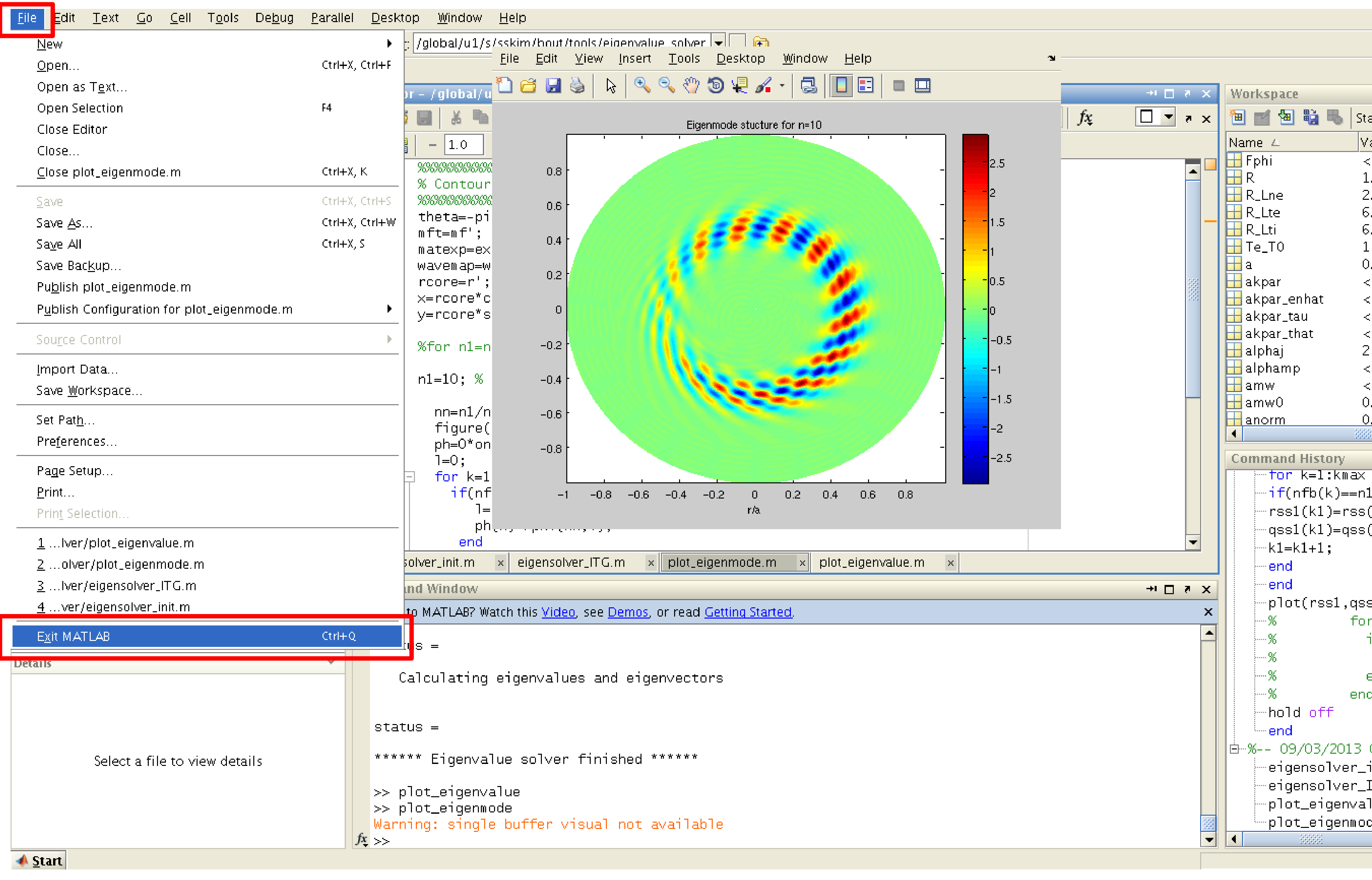
Run plot_eigenmode.m:





File Edit Text Go Cell Tools Debug Parallel Desktop Window Help
Current Folder: /global/u1/s/sskim/bout/tools/eigenvalue_solver
Shortcuts How to Add What's New
Current Folder
tools eigenvalue_solver
Name
eigensolver_init.m
eigensolver_ITG.m
plot_eigenmode.m
plot_eigenvalue.m
README
Details
Select a file to view details
Editor - /global/u
% Contour
theta=-pi
mft=mf';
matexp=ex
wavemap=w
rcore=r';
x=rcore*c
y=rcore*s
%for n1=n
n1=10; %
nn=n1/n
figure(
ph=0*on
l=0;
for k=1
if(nf
l=
ph
end
eigensolver_init.m
eigensolver_ITG.m
plot_eigenmode.m
plot_eigenvalue.m
File Edit View Insert Tools Desktop Window Help
Eigenmode stucture for n=10
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0.2
0.4
0.6
0.8
r/a
2.5
2
1.5
1
0.5
-0.5
-1.5
-2
-2.5
Workspace
Name
Fphi
R
R_Lne
R_Lte
R_Lti
Te_T0
a
akpar
akpar_enhat
akpar_tau
akpar_that
alphaj
alphamp
amw
amw0
anorm
Command History
for k=1:kmax
if(nfb(k)==n1
rss1(k1)=rss(
qss1(k1)=qss(
k1=k1+1;
end
end
plot(rss1,qss
hold off
end
%-- 09/03/2013
Command Window
New to MATLAB? Watch this Video, see Demos, or read Getting Started.
status =
Calculating eigenvalues and eigenvectors
status =
****** Eigenvalue solver finished ******
>> plot_eigenvalue
>> plot_eigenmode
Warning: single buffer visual not available
>>
Start

Exit matlab:


File
Edit
Text
Go
Cell
Tools
Debug
Parallel
Desktop
Window
Help
New
Open... Ctrl+X, Ctrl+F
Open as Text...
Open Selection F4
Close Editor
Close...
Close plot_eigenmode.m Ctrl+X, K
Save Ctrl+X, Ctrl+S
Save As... Ctrl+X, Ctrl+W
Save All Ctrl+X, S
Save Backup...
Publish plot_eigenmode.m
Publish Configuration for plot_eigenmode.m
Source Control
Import Data...
Save Workspace...
Set Path...
Preferences...
Page Setup...
Print...
Print Selection...
1 ...lver/plot_eigenvalue.m
2 ...olver/plot_eigenmode.m
3 ...lver/eigensolver_ITG.m
4 ...ver/eigensolver_init.m
Exit MATLAB Ctrl+Q
Eigenmode stucture for n=10
r/a
Workspace
Command History
Select a file to view details
Calculating eigenvalues and eigenvectors
status =
****** Eigenvalue solver finished ******
>> plot_eigenvalue
>> plot_eigenmode
Warning: single buffer visual not available
Start