Tracing Magnetic Field Lines in BOUT++

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1 Field Line Equations

The equation of a magnetic field line is

$$\frac{d\boldsymbol{R}}{ds} = \frac{\partial \boldsymbol{R}}{\partial x}\frac{dx}{ds} + \frac{\partial \boldsymbol{R}}{\partial y}\frac{dy}{ds} + \frac{\partial \boldsymbol{R}}{\partial z}\frac{dz}{ds} = \frac{\boldsymbol{B}}{|\boldsymbol{B}|}.$$

The magnetic field can be expressed in contravariant form as

$$\boldsymbol{B} = (\boldsymbol{B}\cdot\boldsymbol{\nabla}x)\frac{\partial \boldsymbol{R}}{\partial x} + (\boldsymbol{B}\cdot\boldsymbol{\nabla}y)\frac{\partial \boldsymbol{R}}{\partial y} + (\boldsymbol{B}\cdot\boldsymbol{\nabla}z)\frac{\partial \boldsymbol{R}}{\partial z}.$$

Equating components on both sides of the equation one obtains

$$\frac{dx}{ds} = \frac{\boldsymbol{B}\cdot\boldsymbol{\nabla}x}{|\boldsymbol{B}|}, \quad \frac{dy}{ds} = \frac{\boldsymbol{B}\cdot\boldsymbol{\nabla}y}{|\boldsymbol{B}|}, \quad \frac{dz}{ds} = \frac{\boldsymbol{B}\cdot\boldsymbol{\nabla}z}{|\boldsymbol{B}|}.$$

In BOUT++ the equilibrium field is just $\boldsymbol{B}_0 = \boldsymbol{\nabla}z\times\boldsymbol{\nabla}x$ so that the equations become

$$\frac{dx}{ds} = \frac{\tilde{\boldsymbol{B}}\cdot\boldsymbol{\nabla}x}{\left|\boldsymbol{B}_0+\tilde{\boldsymbol{B}}\right|}, \quad \frac{dy}{ds} = \frac{\left(\boldsymbol{B}_0+\tilde{\boldsymbol{B}}\right)\cdot\boldsymbol{\nabla}y}{\left|\boldsymbol{B}_0+\tilde{\boldsymbol{B}}\right|}, \quad \frac{dz}{ds} = \frac{\tilde{\boldsymbol{B}}\cdot\boldsymbol{\nabla}z}{\left|\boldsymbol{B}_0+\tilde{\boldsymbol{B}}\right|}.$$

To lowest order in $\frac{|\tilde{\boldsymbol{B}}|}{|\boldsymbol{B}_0|}$ these become

$$\frac{dx}{ds} \approx \frac{\tilde{\boldsymbol{B}}\cdot\boldsymbol{\nabla}x}{|\boldsymbol{B}_0|}, \quad \frac{dy}{ds} \approx \frac{\boldsymbol{B}_0\cdot\boldsymbol{\nabla}y}{|\boldsymbol{B}_0|}, \quad \frac{dz}{ds} \approx \frac{\tilde{\boldsymbol{B}}\cdot\boldsymbol{\nabla}z}{|\boldsymbol{B}_0|}.$$

Using the field-aligned coordinate y in lieu of s for integration gives

$$\frac{dx}{dy} = \frac{dx}{ds}\frac{ds}{dy} \approx \frac{\tilde{\boldsymbol{B}}\cdot\boldsymbol{\nabla}x}{\boldsymbol{B}_0\cdot\boldsymbol{\nabla}y}, \quad \frac{dz}{dy} = \frac{dz}{ds}\frac{ds}{dy} \approx \frac{\tilde{\boldsymbol{B}}\cdot\boldsymbol{\nabla}z}{\boldsymbol{B}_0\cdot\boldsymbol{\nabla}y}.$$

The term in the denominator is simply $\boldsymbol{B}_0\cdot\boldsymbol{\nabla}y = (\boldsymbol{\nabla}z\times\boldsymbol{\nabla}x)\cdot\boldsymbol{\nabla}y = \mathcal{J}^{-1} = \frac{B_\theta}{h_\theta}$ so to lowest order in the perturbation amplitude, the field-line equations are:

$$\frac{dx}{dy} \approx \left(\tilde{\boldsymbol{B}}\cdot\boldsymbol{\nabla}x\right)\frac{h_\theta}{B_\theta}, \quad \frac{dz}{dy} \approx \left(\tilde{\boldsymbol{B}}\cdot\boldsymbol{\nabla}z\right)\frac{h_\theta}{B_\theta}.$$

2 Deriving $\tilde{\boldsymbol{B}}$ From $\boldsymbol{\nabla}\times\left(\tilde{A}_\parallel \boldsymbol{b}_0\right)$

First we use a vector identity to express the perturbed magnetic field as

$$\tilde{\boldsymbol{B}} \;=\; \boldsymbol{\nabla}\times\left(\tilde{A}_\parallel\boldsymbol{b}_0\right) = \left(\boldsymbol{\nabla}\tilde{A}_\parallel\right)\times\boldsymbol{b}_0 + \tilde{A}_\parallel\boldsymbol{\nabla}\times\boldsymbol{b}_0. \tag{2.1}$$

Note that in order to remain divergence free, BOTH terms above must be kept:

$$
\begin{aligned}
\boldsymbol{\nabla}\cdot\tilde{\boldsymbol{B}} &= \boldsymbol{\nabla}\cdot\left[\left(\boldsymbol{\nabla}\tilde{A}_\parallel\right)\times\boldsymbol{b}_0\right]+\boldsymbol{\nabla}\cdot\left[\tilde{A}_\parallel\boldsymbol{\nabla}\times\boldsymbol{b}_0\right] \\
&= \left\{\boldsymbol{b}_0\cdot\left[\boldsymbol{\nabla}\times\left(\boldsymbol{\nabla}\tilde{A}_\parallel\right)\right]-\left(\boldsymbol{\nabla}\tilde{A}_\parallel\right)\cdot\nabla\times\boldsymbol{b}_0\right\}+\left\{\tilde{A}_\parallel\boldsymbol{\nabla}\cdot\left(\boldsymbol{\nabla}\times\boldsymbol{b}_0\right)+\left(\boldsymbol{\nabla}\times\boldsymbol{b}_0\right)\cdot\left(\boldsymbol{\nabla}\tilde{A}_\parallel\right)\right\} \\
&= 0
\end{aligned}
$$

The first term in Eq. 1.1 above becomes in covariant form

$$
\begin{aligned}
\left(\boldsymbol{\nabla}\tilde{A}_\parallel\right)\times\boldsymbol{b}_0 &= B_0^{-1}\left(\boldsymbol{\nabla}\tilde{A}_\parallel\right)\times\boldsymbol{B}_0 \\
&= B_0^{-1}\left(\frac{\partial\tilde{A}_\parallel}{\partial x}\boldsymbol{\nabla}x+\frac{\partial\tilde{A}_\parallel}{\partial y}\boldsymbol{\nabla}y+\frac{\partial\tilde{A}_\parallel}{\partial z}\boldsymbol{\nabla}z\right)\times\left(\boldsymbol{\nabla}z\times\boldsymbol{\nabla}x\right) \\
&= B_0^{-1}\left\{\frac{\partial\tilde{A}_\parallel}{\partial x}\boldsymbol{\nabla}x\times\left(\boldsymbol{\nabla}z\times\boldsymbol{\nabla}x\right)+\frac{\partial\tilde{A}_\parallel}{\partial y}\boldsymbol{\nabla}y\times\left(\boldsymbol{\nabla}z\times\boldsymbol{\nabla}x\right)+\frac{\partial\tilde{A}_\parallel}{\partial z}\boldsymbol{\nabla}z\times\left(\boldsymbol{\nabla}z\times\boldsymbol{\nabla}x\right)\right\} \\
&= B_0^{-1}\left\{\begin{array}{c}\frac{\partial\tilde{A}_\parallel}{\partial x}\left[\left(\boldsymbol{\nabla}x\cdot\boldsymbol{\nabla}x\right)\boldsymbol{\nabla}z-\left(\boldsymbol{\nabla}x\cdot\boldsymbol{\nabla}z\right)\boldsymbol{\nabla}x\right]+\frac{\partial\tilde{A}_\parallel}{\partial y}\left[\left(\boldsymbol{\nabla}y\cdot\boldsymbol{\nabla}x\right)\boldsymbol{\nabla}z-\left(\boldsymbol{\nabla}y\cdot\boldsymbol{\nabla}z\right)\boldsymbol{\nabla}x\right] \\ +\frac{\partial\tilde{A}_\parallel}{\partial z}\left[\left(\boldsymbol{\nabla}z\cdot\boldsymbol{\nabla}x\right)\boldsymbol{\nabla}z-\left(\boldsymbol{\nabla}z\cdot\boldsymbol{\nabla}z\right)\boldsymbol{\nabla}x\right]\end{array}\right\} \\
&= B_0^{-1}\left\{\left[g^{11}\frac{\partial\tilde{A}_\parallel}{\partial x}+g^{21}\frac{\partial\tilde{A}_\parallel}{\partial y}+g^{31}\frac{\partial\tilde{A}_\parallel}{\partial z}\right]\boldsymbol{\nabla}z-\left[g^{13}\frac{\partial\tilde{A}_\parallel}{\partial x}+g^{23}\frac{\partial\tilde{A}_\parallel}{\partial y}+g^{33}\frac{\partial\tilde{A}_\parallel}{\partial z}\right]\boldsymbol{\nabla}x\right\} \\
&\equiv f_1\boldsymbol{\nabla}x+f_3\boldsymbol{\nabla}z.
\end{aligned}
$$

The contravariant components are related via $B^i = g^{ij}B_j$ so that in contravariant form this becomes

$$
\left(\boldsymbol{\nabla}\tilde{A}_\parallel\right)\times\boldsymbol{b}_0=\left(g^{11}f_1+g^{13}f_3\right)\frac{\partial\boldsymbol{R}}{\partial x}+\left(g^{21}f_1+g^{23}f_3\right)\frac{\partial\boldsymbol{R}}{\partial y}+\left(g^{31}f_1+g^{33}f_3\right)\frac{\partial\boldsymbol{R}}{\partial z}. \tag{2.2}
$$

The second term in Eq. 1.1 can be manipulated using

$$
\begin{aligned}
\hat{\boldsymbol{b}}_0\times\boldsymbol{\kappa} &= \hat{\boldsymbol{b}}_0\times\left[-\hat{\boldsymbol{b}}_0\times\left(\boldsymbol{\nabla}\times\boldsymbol{b}_0\right)\right] \\
&= \hat{\boldsymbol{b}}_0\cdot\hat{\boldsymbol{b}}_0\left(\boldsymbol{\nabla}\times\boldsymbol{b}_0\right)-\left(\hat{\boldsymbol{b}}_0\cdot\left(\boldsymbol{\nabla}\times\boldsymbol{b}_0\right)\right)\hat{\boldsymbol{b}}_0
\end{aligned}
$$

so that

$$
\boldsymbol{\nabla}\times\boldsymbol{b}_0 = \hat{\boldsymbol{b}}_0\times\boldsymbol{\kappa}+\left(\hat{\boldsymbol{b}}_0\cdot\left(\boldsymbol{\nabla}\times\boldsymbol{b}_0\right)\right)\hat{\boldsymbol{b}}_0.
$$

With this the second term is simply

$$
\tilde{A}_\parallel\boldsymbol{\nabla}\times\boldsymbol{b}_0 = \tilde{A}_\parallel\hat{\boldsymbol{b}}_0\times\boldsymbol{\kappa}+\tilde{A}_\parallel\left(\hat{\boldsymbol{b}}_0\cdot\left(\boldsymbol{\nabla}\times\boldsymbol{b}_0\right)\right)\hat{\boldsymbol{b}}_0.
$$

Expressing this in contravariant form with $\hat{\boldsymbol{b}}_0=\frac{(\boldsymbol{\nabla}z\times\boldsymbol{\nabla}x)\cdot\boldsymbol{\nabla}y}{B_0}\frac{\partial\boldsymbol{R}}{\partial y}=\frac{1}{\mathcal{J}B_0}\frac{\partial\boldsymbol{R}}{\partial y}$ we have

$$
\begin{aligned}
\tilde{A}_\parallel\boldsymbol{\nabla}\times\boldsymbol{b}_0 &= \left[\tilde{A}_\parallel\left(\left(\hat{\boldsymbol{b}}_0\times\boldsymbol{\kappa}\right)\cdot\boldsymbol{\nabla}x\right)\right]\frac{\partial\boldsymbol{R}}{\partial x} \\
&\quad+\left[\tilde{A}_\parallel\left(\left(\hat{\boldsymbol{b}}_0\times\boldsymbol{\kappa}\right)\cdot\boldsymbol{\nabla}y\right)+\frac{\tilde{A}_\parallel}{\mathcal{J}B_0}\left(\hat{\boldsymbol{b}}_0\cdot\left(\boldsymbol{\nabla}\times\boldsymbol{b}_0\right)\right)\right]\frac{\partial\boldsymbol{R}}{\partial y} \\
&\quad+\left[\tilde{A}_\parallel\left(\left(\hat{\boldsymbol{b}}_0\times\boldsymbol{\kappa}\right)\cdot\boldsymbol{\nabla}z\right)\right]\frac{\partial\boldsymbol{R}}{\partial z}.
\end{aligned} \tag{2.3}
$$

Combining Eqs. 1.2 and 1.3, we arrive at an expression for $\tilde{\boldsymbol{B}}$ in contravariant form:

$$
\begin{aligned}
\tilde{\boldsymbol{B}} &= \left\{g^{11} f_1 + g^{13} f_3 + \tilde{A}_\parallel \left(\left(\hat{\boldsymbol{b}}_0 \times \boldsymbol{\kappa}\right) \cdot \boldsymbol{\nabla} x\right)\right\} \frac{\partial \boldsymbol{R}}{\partial x} \\
&\quad + \left\{g^{21} f_1 + g^{23} f_3 + \tilde{A}_\parallel \left(\left(\hat{\boldsymbol{b}}_0 \times \boldsymbol{\kappa}\right) \cdot \boldsymbol{\nabla} y\right) + \frac{\tilde{A}_\parallel}{\mathcal{J} B_0} \left(\hat{\boldsymbol{b}}_0 \cdot (\boldsymbol{\nabla} \times \boldsymbol{b}_0)\right)\right\} \frac{\partial \boldsymbol{R}}{\partial y} \\
&\quad + \left\{g^{31} f_1 + g^{33} f_3 + \tilde{A}_\parallel \left(\left(\hat{\boldsymbol{b}}_0 \times \boldsymbol{\kappa}\right) \cdot \boldsymbol{\nabla} z\right)\right\} \frac{\partial \boldsymbol{R}}{\partial z}.
\end{aligned} \tag{2.4}
$$

3 Lowest Order Line Tracing Equations

The line-tracing equation depends upon $\tilde{\boldsymbol{B}} \cdot \boldsymbol{\nabla} x$, $\tilde{\boldsymbol{B}} \cdot \boldsymbol{\nabla} y$, and $\tilde{\boldsymbol{B}} \cdot \boldsymbol{\nabla} z$. These can be read off from above:

$$
\tilde{\boldsymbol{B}} \cdot \boldsymbol{\nabla} x = g^{11} f_1 + g^{13} f_3 + \tilde{A}_\parallel \left(\left(\hat{\boldsymbol{b}}_0 \times \boldsymbol{\kappa}\right) \cdot \boldsymbol{\nabla} x\right) \tag{3.1}
$$

$$
\tilde{\boldsymbol{B}} \cdot \boldsymbol{\nabla} y = g^{21} f_1 + g^{23} f_3 + \tilde{A}_\parallel \left(\left(\hat{\boldsymbol{b}}_0 \times \boldsymbol{\kappa}\right) \cdot \boldsymbol{\nabla} y\right) + \frac{\tilde{A}_\parallel}{\mathcal{J} B_0} \left(\hat{\boldsymbol{b}}_0 \cdot (\boldsymbol{\nabla} \times \boldsymbol{b}_0)\right) \tag{3.2}
$$

$$
\tilde{\boldsymbol{B}} \cdot \boldsymbol{\nabla} z = g^{31} f_1 + g^{33} f_3 + \tilde{A}_\parallel \left(\left(\hat{\boldsymbol{b}}_0 \times \boldsymbol{\kappa}\right) \cdot \boldsymbol{\nabla} z\right) \tag{3.3}
$$

where

$$
\begin{aligned}
g^{11} &= (R B_\theta)^2 \\
g^{12} &= 0 \\
g^{13} &= -I \,(R B_\theta)^2 \\
g^{22} &= 1/h_\theta^2 \\
g^{23} &= \nu/h_\theta^2 \\
g^{33} &= I^2 \,(R B_\theta)^2 + B_0^2/\,(R B_\theta)^2
\end{aligned}
$$

and

$$
\begin{aligned}
f_1 &= -B_0^{-1} \left[g^{13} \frac{\partial \tilde{A}_\parallel}{\partial x} + g^{23} \frac{\partial \tilde{A}_\parallel}{\partial y} + g^{33} \frac{\partial \tilde{A}_\parallel}{\partial z}\right] \\
f_3 &= B_0^{-1} \left[g^{11} \frac{\partial \tilde{A}_\parallel}{\partial x} + g^{21} \frac{\partial \tilde{A}_\parallel}{\partial y} + g^{31} \frac{\partial \tilde{A}_\parallel}{\partial z}\right].
\end{aligned}
$$

Line Tracing Equations The field line equations, to lowest order in $\frac{|\tilde{\boldsymbol{B}}|}{|\boldsymbol{B}_0|}$, are given by

$$
\frac{dx}{dy} \cong \left[g^{11} f_1 + g^{13} f_3 + \tilde{A}_\parallel \left(\left(\hat{\boldsymbol{b}}_0 \times \boldsymbol{\kappa}\right) \cdot \boldsymbol{\nabla} x\right)\right] \frac{h_\theta}{B_\theta} \tag{3.4}
$$

$$
\frac{dz}{dy} \cong \left[g^{31} f_1 + g^{33} f_3 + \tilde{A}_\parallel \left(\left(\hat{\boldsymbol{b}}_0 \times \boldsymbol{\kappa}\right) \cdot \boldsymbol{\nabla} z\right)\right] \frac{h_\theta}{B_\theta}. \tag{3.5}
$$