Simple examples on finding instantaneous frequency using Hilbert transform:

As the name implies, instantaneous frequency finds the frequency at the present instant. In this post we will demonstrate it using two simple toy examples. We will consider simulated amplitude modulated signal and frequency modulated signal. Though these examples are elementary, modulation is a commonly occurring phenomenon in real machinery signals.

Brief Introduction to Hilbert transform:

Hilbert transform is a time domain transformation defined as

$$\widetilde{x}(t) = H[x(t)] = \int_{-\infty}^{\infty} \frac{x(u)}{\pi(t-u)} du$$

Using the definition of convolution, it can be represented as convolution of $x(t)$ with $\frac{1}{\pi t}$. So

$$\widetilde{x}(t) = H[x(t)] = x(t) * \frac{1}{\pi t}$$

Convolution plays an important role in the actual computation of Hilbert transform. Omitting proofs, though proofs are simple, we will state some facts below.

- Actual computation of Hilbert transform is simpler in frequency domain than in time domain. Fourier transform of convolution of two functions in time domain is equivalent to product of Fourier transforms in the frequency domain. Representing Fourier transform by $\mathscr{F}$, it is a known result that $\mathscr{F}[\frac{1}{\pi t}] = -j \times sign(f)$. And it is convenient to obtain Fourier transform of any time domain signal. So to obtain the signal's Hilbert transform we have to only multiply the positive frequencies of the signal by $(-j)$ and negative frequencies by (*j*) and remove the zero frequency term (i.e. the constant term). Finally by transforming back the result to time domain, by IFFT, we get the desired Hilbert transform. This procedure will give us the Hilbert transform of any arbitrary signal. Note that the Hilbert transform of a time domain signal is another time domain signal.

Analytic signal representation:

Analytic signals are those that contain only positive frequency components. It has several applications, one of which is to obtain instantaneous frequency. By using Hilbert transform, we can obtain analytic signal representation of a signal. Representing the analytic signal by $x_a(t)$ we have

$$x_a(t) = x(t) + j\widetilde{x}(t)$$

Analytic signal representation is absolutely central to obtain instantaneous frequency content of the signal. This follows from the polar representation of the analytic signal. In polar form,

$$x_a(t) = A(t)e^{j\phi(t)}$$

where

$$A(t) = \sqrt{(x(t)^2 + \widetilde{x}(t)^2} \qquad \text{and} \qquad \phi(t) = atan\left(\frac{\widetilde{x}(t)}{x(t)}\right)$$

Instantaneous frequency is obtained by differentiating phase with respect to time. Representing instantaneous frequency by $f(t)$,

$$f(t) = \frac{1}{2\pi}\frac{d\phi(t)}{dt}$$

The factor $\frac{1}{2\pi}$ is used to convert radian into Hz.

Now that we have known how to obtain instantaneous frequency, we will use two toy examples to show it. We will use amplitude modulation and frequency modulation signals to demonstrate it.

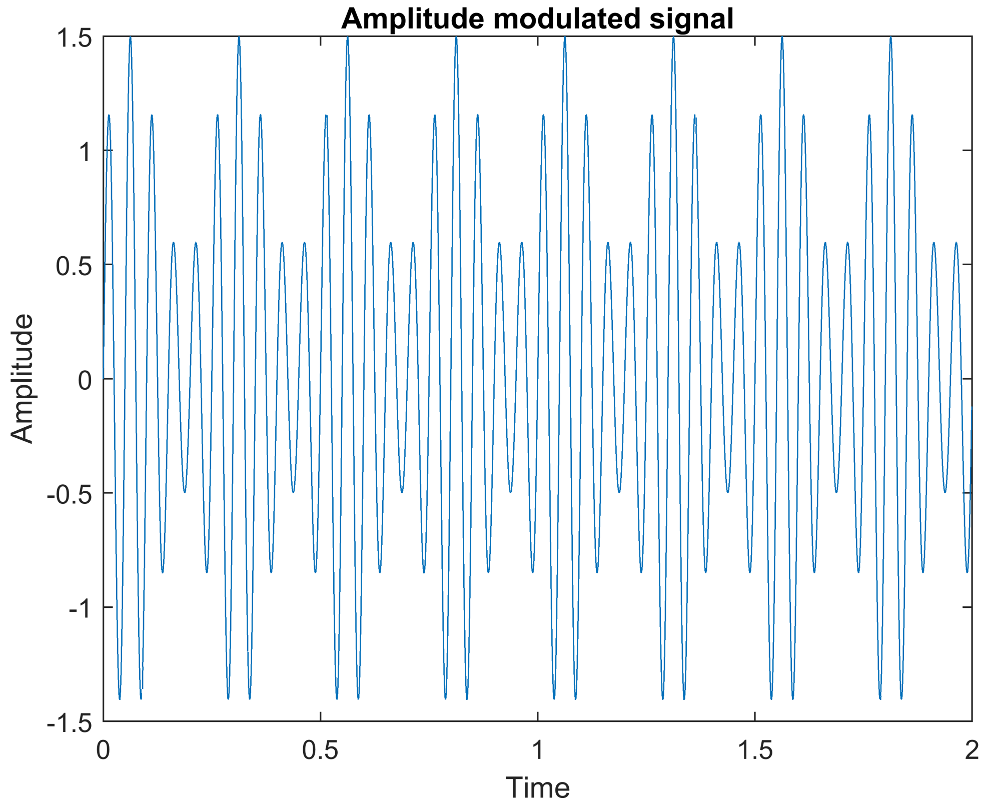
Amplitude Modulation:

Mathematically, an amplitude modulated signal can be represented as

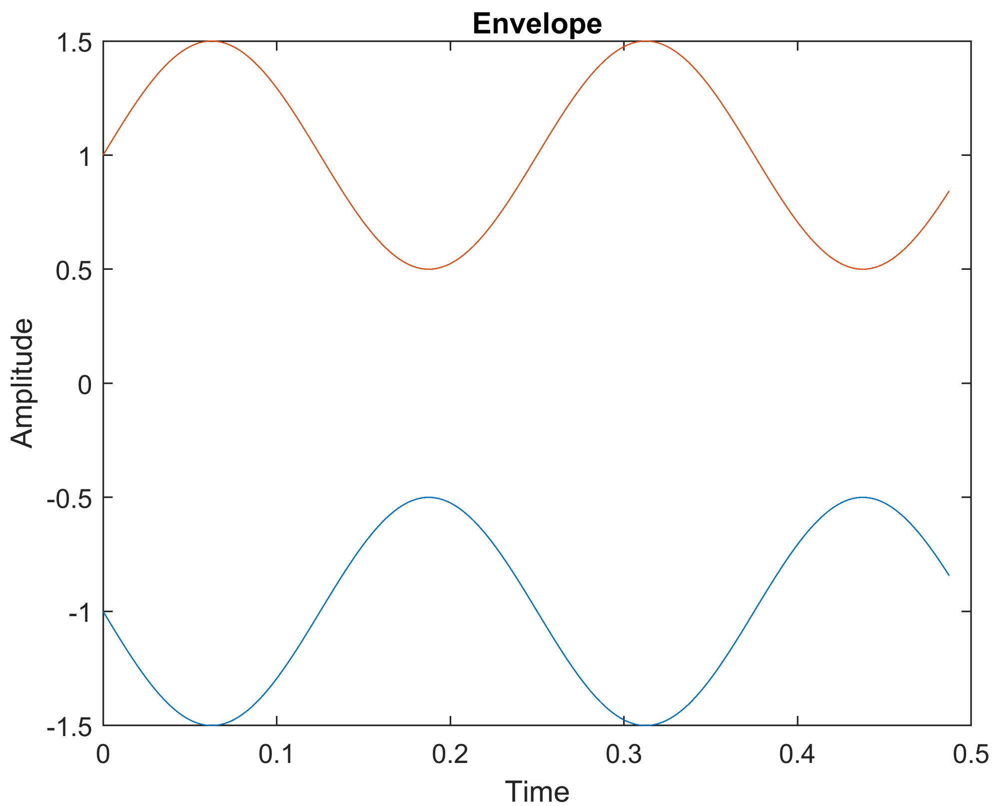
$$y(t) = (A + Bsin\,(2\pi f_m t))\sin\,(2\pi f_c t) = A\left(1 + \left(\frac{B}{A}\right)\sin\,(2\pi f_m t)\right)\sin\,(2\pi f_c t)$$

where, f_m is the message frequency, f_c is the carrier frequency, A and B are amplitudes and $\left(\frac{B}{A}\right)$ is the modulation index. Taking f_m= 20 Hz, f_c= 4 Hz, Sampling frequency = 1024 Hz and time period (T) = 1 sec, we plot the following results.

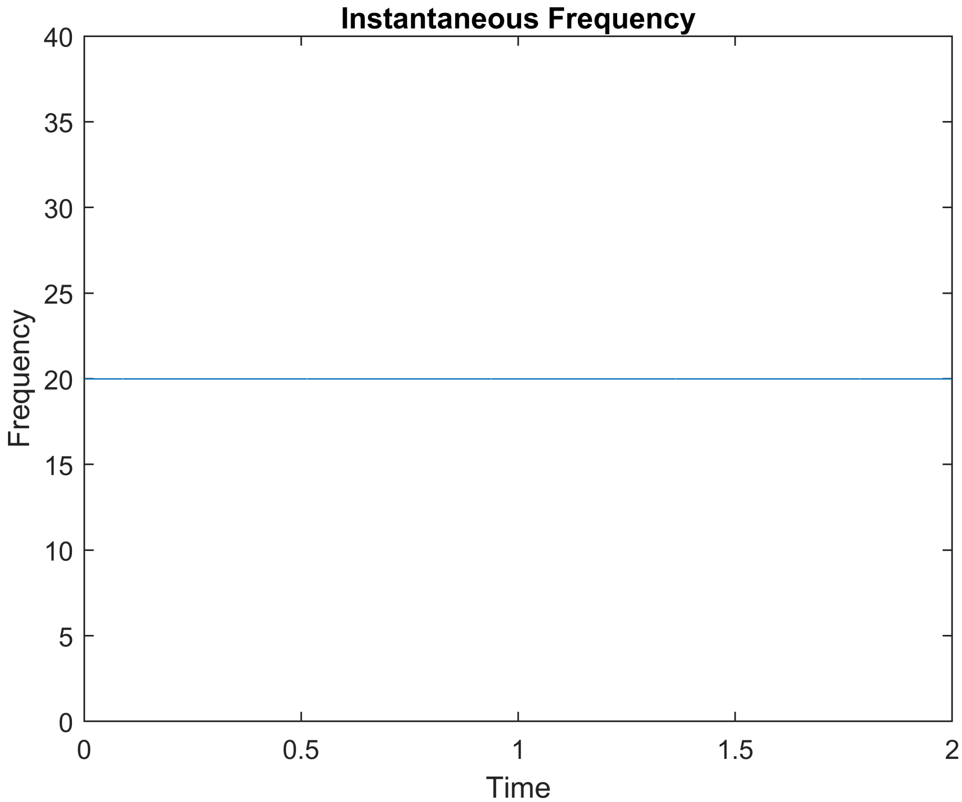
```
fc = 20;
fm = 4;
fs = 1024;
t = 0:1/fs:2-1/fs;
N = length(t);
yc = 2*sin(2*pi*fc*t);
y = 1*(1+.5*sin(2*pi*fm*t)).*sin(2*pi*fc*t);
% Plot amplitude modulated signal
plot(t,y);title("Amplitude modulated signal");xlabel("Time");
xlabel("Time");ylabel("Amplitude")
```



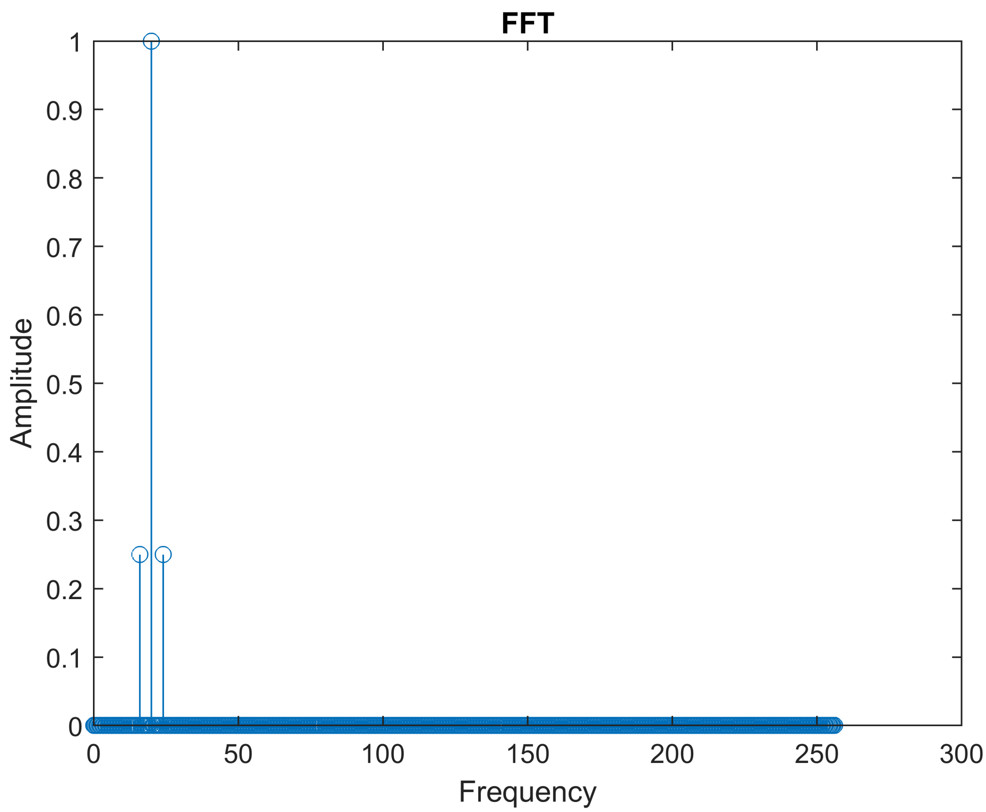

```
% Generate analytic signal (y1)
y1 = hilbert(y);
% Plot envelope of the signal
y2 = abs(y1);
figure
plot(t(1:500),[-1;1]*y2(1:500)); title("Envelope");
xlabel("Time");ylabel("Amplitude")
```




```
% Find instantaneous frequency
instfreq = fs/(2*pi)*diff(unwrap(angle(y1)));
% Plot instantaneous frequency. Note that instantaneous frequency is
% constant in this case.
figure
plot(t(2:end),instfreq); title("Instantaneous Frequency");
xlabel("Time"); ylabel("Frequency")
axis([0 2 0 40])
```




```
% FFT of apmplitude modulated signal
z = fft(y);
f = (fs/N)*(0:fs/2);
z1 = (2/N)*abs(z);
figure
stem(f,z1(1:fs/2+1)); title("FFT");xlabel("Frequency");ylabel("Amplitude")
```



Frequency Modulated Signal:

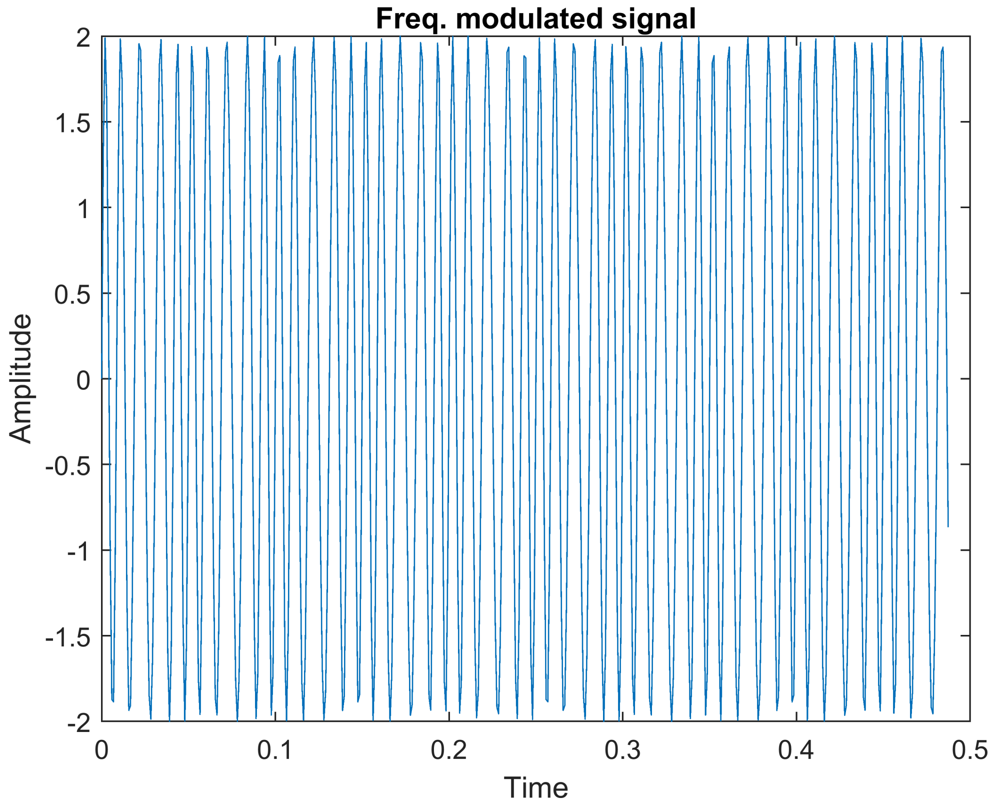
According to Wikipedia, frequency modulated signal can be represented as

$$y(t) = A_c \cos\left(2\pi f_c t + \left(\frac{A_m f_\Delta}{f_m}\right) sin(2\pi f_m t)\right)$$

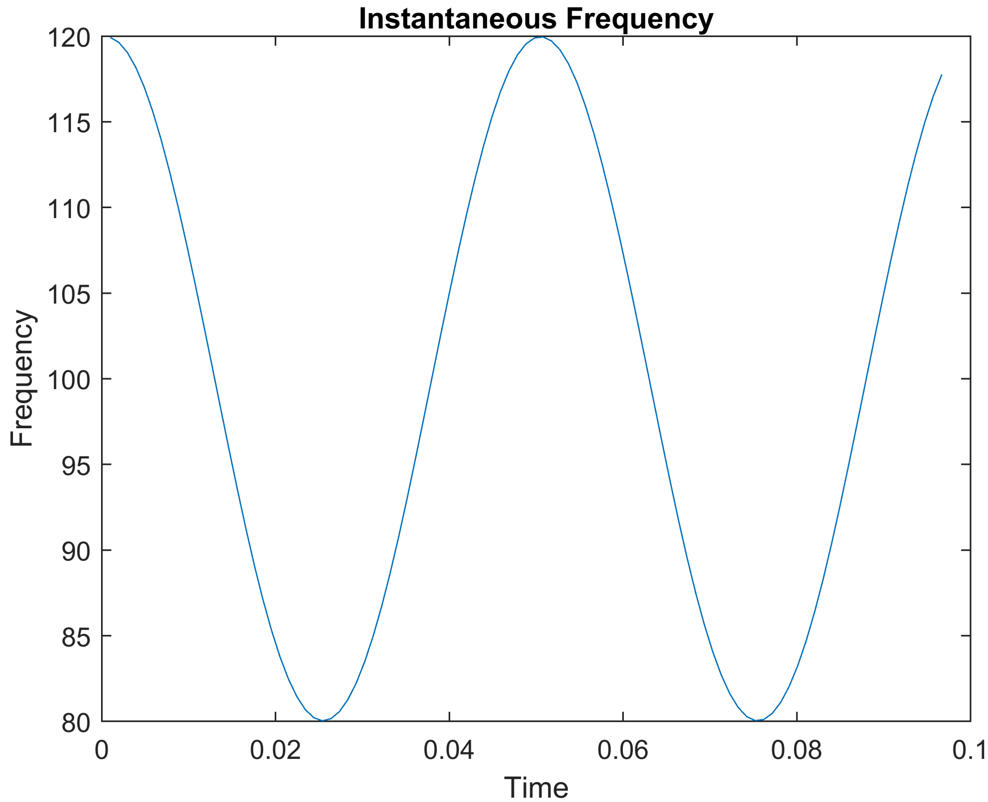
Where, $\left(\frac{A_m f_\Delta}{f_m}\right) = \beta$ is the modulation index.

Taking f_m=20 Hz, f_c= 100 Hz, sampling frequency = 1024 Hz, time duration (T) = 1 sec, we plot the following results.

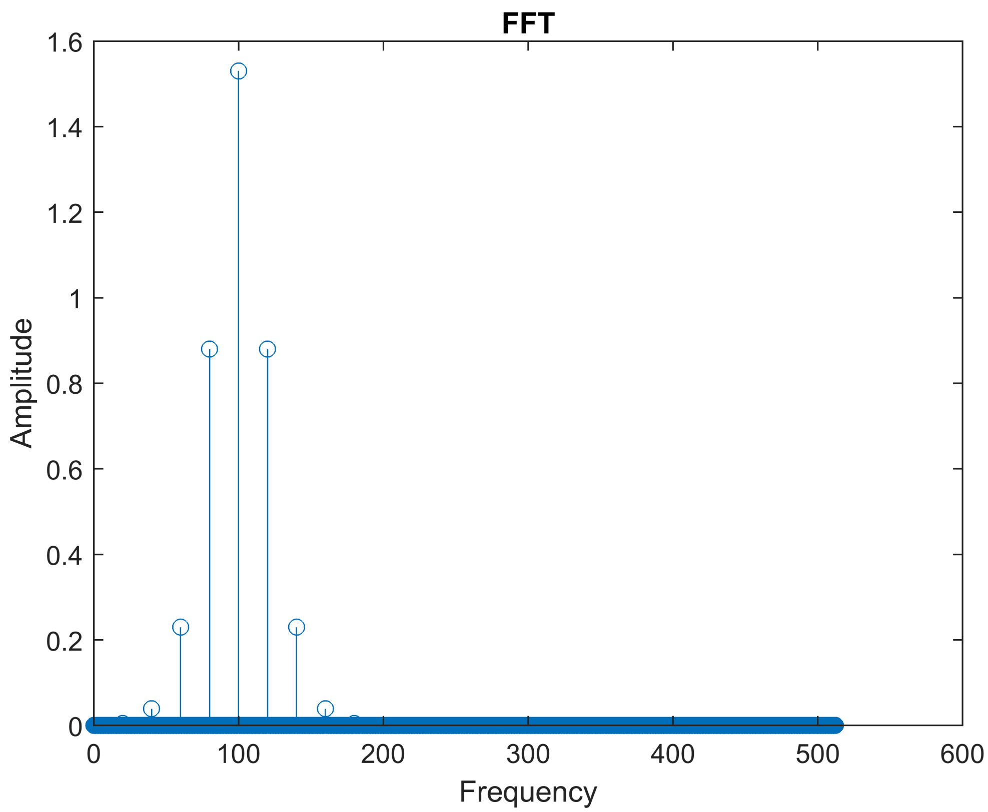
```
fc = 100;
fm = 20;
beta = 1;  % modulation index
fs = 1024;
t = 0:1/fs:2-1/fs;
N = length(t);
ym = sin(2*pi*fm*t);
y = 2*sin(2*pi*fc*t+beta*sin(2*pi*fm*t));
figure
plot(t(1:500),y(1:500)) % Plot of original modulated signal
title("Freq. modulated signal"); xlabel("Time");ylabel("Amplitude")
```



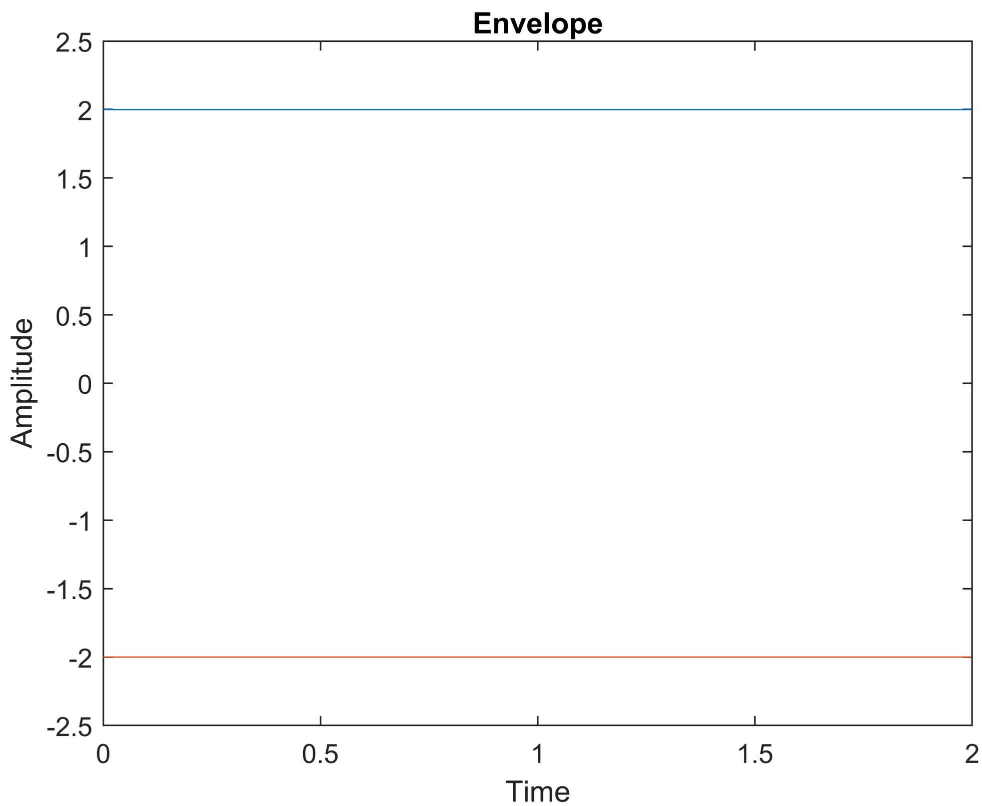

```
y1 = hilbert(y);
y2 = unwrap(angle(y1));
instfreq = (fs/(2*pi))*diff(y2);
figure
plot(t(2:100),instfreq(1:99)) % Plot of instantaneous freq. signal
title("Instantaneous Frequency");xlabel("Time");ylabel("Frequency")
```




```
figure
z = fft(y);
f = (fs/N)*(0:N/2);
z1 = (2/N)*abs(z);
stem(f,z1(1:N/2+1)) % FFT of frequency modulated signal
title("FFT");xlabel("Frequency");ylabel("Amplitude")
```




```
% The envelopes are straight lines as the amplitude doesn't change.
figure
plot(t,[1;-1]*abs(y1)); title("Envelope");
xlabel("Time");ylabel("Amplitude")
```



Reference:

An excellent reference from which almost all of the theory sections of this post have been taken is:

1. Thrane, N. "The Hilbert Transform." *B&K Technical Rewiew* 3 (1984).

Last modified: 11^{th} June, 2019.